CLASSIFICATION OF LOW DIMENSIONAL SUSY-EXTENSIONS OF CW-SPACES

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Abstract. We will give here a brief summary of the classification result from our habilitation thesis *Supersymmetric extensions of solvable Lorentzian symmetric spaces*, 2013.

Introduction and Summary

Introduction.[1] Supersymmetry can be seen as an extension of the set of isometries of a given manifold in a natural way. It has been identified in the 1970th as a fundamental symmetry to overcome the obstructions due to the Coleman-Mandula Theorem. The mentioned Theorem states that some basic physical predictions force the symmetry algebra of a field theory to be a trivial extension of the Poincaré-algebra $p(3,1) := \mathfrak{so}(3,1) \ltimes \mathbb{R}^{3,1}$. I.e. any additional fields that are in a representation of some internal symmetry group G are acted upon trivially by $p(3,1)$ such that the full symmetry algebra is a direct product $p(3,1) \times \mathfrak{g}$, see [14].

The authors in [23] dropped one of the basic assumptions and proved that there exists indeed a non-trivial extension of $p(3,1)$ such that the generators mix non-trivially. For this they need to allow algebras that are no longer Lie algebras but super Lie algebras. Such algebras admit generators that obey anti-commutation instead of commutation relations. They proved that a minimal extension has to be of the form $p(3,1) \times S$ with S being a spin representation of $\mathfrak{so}(3,1)$ and $\mathbb{R}^{3,1}$ acting trivially. Furthermore, the bracket of two fermionic generators close into $\mathbb{R}^{3,1}$ and some kind of super Jacobi identity has to be satisfied such that we ends up with the so called super Poincaré algebra. This Theorem is known as the Haag-Lopuszanski-Sohnius Theorem. With some effort this can be extended to various dimensions, provided there exists some Spin-invariant symmetric bilinear

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[1] The numbering in the Introduction refers to the main text of our thesis. This will shortly be fixed.

map on S that closes into the vectors. Such bilinears are fully classified, see [1, 24] or [12].

Coming from this flat situation its the next step to look for superalgebras that generalize the super Poincaré algebra for curved manifolds. There are two different points of view. On the one hand, we can look for such super algebra as a subalgebra of the vector fields of certain super manifolds; see for example [2, 27, 29] for a representation on graded manifolds and their deformed variants, or the nice books [4, 22] for a general discussion of supersymmetry in superspace formulation, and [15] for the particular dimension eleven. On the other hand, we can stay more close to the flat situation and directly ask if an extension of the isometries to a super Lie algebra exists. Here we will follow the second approach.

The class of manifolds that we concentrate on in our text are solvable Lorentzian symmetric spaces. They have been introduced in [10] and thus a frequently used name for such spaces is Cahen-Wallach spaces. Other names that are also common in physics literature are KG-spaces or *pp*-waves, although they only form a special class among them. For the description and classification of solvable symmetric spaces of signature greater than one, see [10, 37] or the nice cohomological description in [25]. Although the construction of Cahen-Wallach spaces at first glance seems to be not very involved, their applications are rich. From a differential geometrical point of view they play a role in the classification of holonomy groups of Lorentzian manifolds, see [8, 34].

In view of our text, both terms – supersymmetry and solvable Lorentzian symmetric space – are considered together. In general, manifolds that allow supersymmetry are used as basic candidates for supergravity backgrounds and therefore are the natural objects in high energy physics. Among these, homogeneous spaces – and in particular the special kind of manifolds considered here – often appear in this context, because they have a large space of isometries and admit many supersymmetry generators, see for example [13, 19, 17, 20, 21, 36]. A supersymmetry generator of a supergravity background M is basically a section ξ in a bundle S over M that satisfies a first order differential equation $D\xi = \nabla\xi + \Omega\xi = 0$ with Ω being a one-form with values in the endomorphism of S. In this context S is the spinor bundle or some twisted version of it and D is a connection on this bundle. The spinors with $D\xi = 0$ are usually called Killing spinors, although the term classically describes a more special situation with $\Omega(X)\xi \sim X\xi$, see [5, 6], for example.

There is some further structure that is necessary for supergravity. Firstly, the Killing spinors have to be compatible with the isometries represented by the Killing vector fields and, secondly, two Killing spinors must be able to be combined to a Killing vector field. Last but not least, Killing vector fields and Killing spinors have to form a super Lie algebra. The approach in the cited literature to get manifolds that allow supersymmetry usually is local and the fields that enter into the discussion are limited by construction. On the one hand, we use the homogeneous structure of the manifold to overcome the local description and use a more algebraic one. Nevertheless, using a particular frame, for example coming from the Maurer-Cartan form, would provide us with the possibility to "localize" our results, see [10] for this special situation and [3] for a more general discussion. On the other hand, we forget about supergravity that might limit the form of the fields and the dimension of interest. We start with the following question: What connections on a solvable Lorentzian symmetric space allow supersymmetry? First attempts toward this question have been made in [39], though with a very restrictive constraint on the closing of the algebra. Nevertheless, the author's description of representation of the Heisenberg algebra on spinor modules turn out to be useful for the first part of our text, in particular Section 3.

The answer of the question is addressed in several steps that also reflect the organization of our text. We begin by considering a quadratic relation on pairs of elements of the complex Clifford algebra of a real vector space. We give several solutions of this relation and discuss them in detail, see Propositions 1.5 and 1.8 with Table 1 and Propositions 1.21 and 1.23. In fact, the examples we present are complete in a sense that will become clear later in the text. After shortly recalling the notion of solvable Lorentzian symmetric spaces and its construction in Section 2 we turn to the first important ingredient of the construction of superalgebras, the so called Clifford maps and Clifford representations, in Section 3. We first give a general definition and discuss the relation of our notion to special connections on homogeneous spaces. Although Clifford maps have a very geometric meaning we prefer our notion due to the algebraic nature of our approach. Next we turn to the manifolds of interest and classify their irreducible Clifford maps and Clifford representations. The result is collected in Theorems 3.9 and 3.10. In particular, in the most general applicable situation of Theorem 3.9 the maps depend on pairs that are related to the ones discussed in the first section.

In Section 4 we turn to the main part of our text, i.e. to the systematic description and classification of solvable Lorentzian symmetric spaces that allow supersymmetry. We start with Definition 4.1 which is formulated for general homogeneous spaces. One major ingredient is a bilinear map on a spinor bundle that takes its values in a Lie algebra that is closely related to the ones that come from the underlying homogeneous structure. The bilinear map fulfills a compatibility condition that turns the Lie algebra into a super algebra or a super Lie algebra, see Definition 4.1(iv) and (v). Moreover, we specify one part of this bilinear map, see Definition 4.4. In Section 4.2 we again turn to our special class of manifolds and describe in detail the conditions on the bilinear map such that a super extension of a solvable Lorentzian symmetric space exists. We do this for the irreducible situations according to Theorems 3.9 and 3.10 in Section 4.2.1 and 4.2.2, separately. The results are collected in Theorems 4.14 and 4.24, respectively. The subsequent Section 5 addresses the question whether such a super extension is a SUSY extension. We give the general results in Theorems 5.4 and 5.6. We use the results to classify all irreducible SUSY extensions of solvable Lorentzian symmetric spaces in Sections 5.2 and 5.3. The results are collected in Tables 2 and 3.

The odd part of the super extension is allowed to be reducible in Section 6. This has been denoted by N-SUSY extension in Definition 4.1. After a general discussion we consider a particular situation that allows to carry over the tools we developed before. In particular, we are in the position to fill gaps in the low dimensional classification of the preceding section, see Table 4. Although we started the reducible case to fill the gaps in low dimensions, our construction may yield further SUSY extensions in any case. In Table 5 we give a list of 2-SUSY extension. It particular, Tables 2, 4, and 5 together give a complete list of natural, flat N-extensions up to dimension eleven for $N \leq 2$. An additional feature that is present in these low dimensions is a restriction on the eigenvalue structure of the symmetric map B that enters into the construction of the solvable Lorentzian symmetric space, see Theorem 6.14. Due to the structure of the extensions there is a close relation to the irreducible situation that is described in Section 6.3. This leads to a way to obtain N-SUSY extensions from irreducible super extensions by a procedure we call truncation. In this context we also discuss the case where the super extension is itself a SUSY extension, such that the two algebras may be considered as reduction or oxidation of each other.

Summary. In this text we systematically discuss supersymmetry on solvable Lorentzian symmetric spaces. We provide conditions for spinor connections to allow supersymmetry by explicitly describing the bracket structure. We develop in detail the necessary notions and tools for flat connections. With that we show that in any dimension up to eleven non-trivial, at most canonically restricted, irreducible supersymmetry exists and we extend the list of examples that can be found in the literature to a full classification. We also extend this classification by a complete list of $N = 2$-extended natural supersymmetries.

By presenting further examples – for $N > 2$, non-natural, non canonically restricted, or non-flat – we show how our tools can be applied to the general situation. In particular, we show that in any dimension that allows an irreducible super extension a non-trivial $^3/_4$-restricted supersymmetry is possible.

Brief Summary of the Classification

Definition 1. Let (V, g) be a Euclidean vector space and $C\ell(V)$ as above. For $a, b \in C\ell(V)$ consider the map $q_{a,b} : C\ell(V) \to C\ell(V)$ that is defined by

$$q_{a,b}(x) := a^2 x + x b^2 - 2axb\,. \tag{1}$$

A *quadratic Clifford pair* is a pair $(a, b) \in C\ell(V) \times C\ell(V)$ that obeys

(i) $q_{a,b}(V) \subset V$ and
(ii) $q_{a,b}$ restricted to V is g-symmetric, i.e. $q_{a,b}\big|_V \in \mathrm{End}(V)$.

We call the quadratic Clifford pair *(non)-degenerate* if $q_{a,b}$ is a (non)-degenerate symmetric map and we call it *associated to* $B \in \mathrm{End}(V)$ if $q_{a,b}\big|_V = B$.

For $a, b \in C\ell(V)$ the map $q_{a,b}$ is the square of the map $s_{a,b}$ with $s_{a,b}(x) := ax - xb$.

Proposition 2. *Two monomials $c = \alpha\Gamma_I$, $d = \beta\Gamma_J$ yield a non-degenerate quadratic Clifford pair if and only if the index sets I and J coincide up to order or one of both is empty. If I and J coincide up to order and $\hat{I} > 0$, the eigenvalues of $q_{c,d}$ are up to sign given by $(\alpha \pm \beta)^2$ with multiplicities $\hat{I}$ and $\dim V - \hat{I}$.*

Proposition 3. *For a particular choice of parameters the approach $c = (\alpha\Gamma_I + \beta\Gamma_J)\Gamma_K$, $d = (\gamma\Gamma_I + \delta\Gamma_J)\Gamma_K$ with non-vanishing scalars $\alpha, \beta, \gamma, \delta$ and $I \cap J = I \cap K = J \cap K = \emptyset$ yields non-monomial quadratic Clifford pairs. Moreover, $B \not\sim \mathbb{1}$ and non-degenerate if and only if the combination is from Table 1.*

Table 1. Non-monomial, non-degenerate quadratic Clifford pairs with $B \not\sim \mathbb{1}$

Pair	Eigenvalues
$((\alpha + \beta\Gamma^*)\Gamma_I, (\alpha + \beta\Gamma^*)\Gamma_I)$	$\begin{cases} 4\sigma_I\alpha^2\,, & \hat{I} \text{ even} \\ 4\sigma_I\beta^2\,, & \dim V - \hat{I} \text{ even} \end{cases}$
$((\alpha + \beta\Gamma^*)\Gamma_I, -(\alpha + \beta\Gamma^*)\Gamma_I)$	$\begin{cases} 4\sigma_I\beta^2\,, & \hat{I} \text{ even} \\ 4\sigma_I\alpha^2\,, & \dim V - \hat{I} \text{ even} \end{cases}$
$(\alpha(\cos(\phi) + \sin(\phi)\Gamma^*)\Gamma_I,$ $\beta(\cos(\phi) - \sin(\phi)\Gamma^*)\Gamma_I)$	$\begin{cases} \sigma_I(\alpha - \beta)^2\cos(2\phi)\,, & \hat{I} \text{ odd} \\ \sigma_I(\alpha + \beta)^2\cos(2\phi)\,, & \dim V - \hat{I} \text{ odd} \end{cases}$

Definition 4. The pairs from Table 1 are called *pseudo-monomial quadratic Clifford pairs of even* and *odd type*, respectively.

[1] We use in our summary of the classification a consistent numbering that is different from the one in the thesis and the introduction.

Definition 5. Consider the square root of $q_{a,b}$ given by $s_{a,b} : x \mapsto ax - xb$. We call a pair $(a,b) \in \mathrm{C}\ell(V)$ linear if $s_{a,b}$ restricted to $V \otimes \mathbb{C}$ has image in $V \otimes \mathbb{C}$.

Proposition 6. *A quadratic Clifford pair is linear if and only if it is of the form* $(0,\beta)$ *with a scalar* β *or* (A,A) *with* $A = A_0 + iA_1 \in \mathrm{C}\ell_2(V)$ *such that* A_0 *and* A_1 *anti-commute when considered as endomorphisms of* V*.*

Proposition 7. *Consider* $c = \sum_\alpha c_\alpha \Gamma_{I_\alpha}$ *and* $d = \sum_\alpha d_\alpha \Gamma_{I_\alpha}$ *with* $d_\alpha = \epsilon_\alpha c_\alpha$ *for* $\epsilon_\alpha \in \{\pm 1\}$, $I_\alpha \cap I_\beta = \emptyset$ *for* $\alpha \neq \beta$, *and* $\big(\bigcup_\alpha I_\alpha\big)^{\complement} = \emptyset$. *Furthermore, let the amount of summands in* c *be bigger than two. Then* (c,d) *is a non-degenerate quadratic Clifford pair if and only if* $\hat{I}_\alpha$ *is even for all* α *in even dimension and* $\hat{I}_\alpha$ *is even up to one exception in odd dimension. Furthermore we have* $\epsilon_\alpha = (-1)^{\hat{I}_\alpha}$. *In this situation the eigenvalues of* $q_{c,d}$ *are given by* $4\sigma_{I_\alpha} c_\alpha^2$ *with multiplicities* $\hat{I}_\alpha$.

Proposition 8. *Consider* $c = \sum_\alpha c_\alpha \Gamma_{I_\alpha} + \Gamma^* \sum_\alpha \hat{c}_\alpha \Gamma_{I_\alpha}$ *and* $d = \sum_\alpha \epsilon_\alpha c_\alpha \Gamma_{I_\alpha} + \Gamma^* \sum_\alpha \hat{\epsilon}_\alpha \hat{c}_\alpha \Gamma_{I_\alpha}$ *with* $\epsilon_\alpha = -\hat{\epsilon}_\alpha = (-1)^{\hat{I}_\alpha}$. *Then* (c,d) *yields a non-degenerate quadratic Clifford pair if and only if it is of one of the following types:*

(1) All summands are even with $\hat{c}_\alpha c_\alpha = 0$ *for all* α, *i.e. only one of* Γ_{I_α} *and* $\Gamma^*\Gamma_{I_\alpha}$ *contributes to* c *and* d.

(2) Two summands are odd, e.g. $\hat{I}_{\alpha_0}, \hat{I}_{\alpha_1}$, *and all others are even with* $\hat{c}_\alpha = 0$ *as well as* $c_{\alpha_0} c_{\alpha_1} = \hat{c}_{\alpha_0}\hat{c}_{\alpha_1}$. *For example:*

$$
\begin{aligned}
(c &= c_{\alpha_0}(\gamma + \Gamma^*)\Gamma_{I_{\alpha_0}} + c_{\alpha_1}(1 + \gamma\Gamma^*)\Gamma_{I_{\alpha_1}} + \sum_{\hat{I}_\alpha \text{ even}} c_\alpha \Gamma_{I_\alpha}\,, \\
d &= -c_{\alpha_0}(\gamma - \Gamma^*)\Gamma_{I_{\alpha_0}} - c_{\alpha_1}(1 - \gamma\Gamma^*)\Gamma_{I_{\alpha_1}} + \sum_{\hat{I}_\alpha \text{ even}} c_\alpha \Gamma_{I_\alpha})
\end{aligned}
$$

if $\hat{c}_0, c_1 \neq 0$.

Definition 9. The quadratic Clifford pairs of Propositions 7 and 8 are called *generalized monomial*. We denote them by the sequence of the lengths that uniquely determine their type, e.g. $(2,4,6)$, $(3,2,2,6)$, or $(2,\bar{2},4,\bar{6})$, $(3,5,2,6)$. Here $\bar{2}$ denotes a contribution in case 1 which come with Γ^*.

Proposition 10 ([10])**.** *Solvable Lorentzian symmetric spaces are in one-to-one correspondence with triples* $(V, \langle\cdot,\cdot\rangle, B)$ *with*

- V *is an* n*-dimensional real vector space,*
- $\langle\cdot,\cdot\rangle$ *is a block-diagonal Lorentzian metric on* $W := V \oplus \mathbb{R}^2$ *such that* $g := \langle\cdot,\cdot\rangle\big|_V$ *is positive definite,*
- $B \in \mathrm{End}(V)$ *is symmetric with respect to* g

$\mathfrak{g} = V^* \oplus W = V^* \oplus (V \oplus \mathbb{R}_+ \oplus \mathbb{R}_-)$ *is a Lie algebra subject to the commutation relations*

$$[v^*, w] = -v^*(Bw)\cdot e_+ = -\langle Bv, w\rangle \cdot e_+\,, \tag{2}$$

$$[v^*, e_-] = Bv\,, \tag{3}$$
$$[e_-, w] = w^*\,, \tag{4}$$

and all other combinations vanishing. The associated symmetric space is denoted by M_B.

- $\langle\cdot,\cdot\rangle$ *is extended to a bi-invariant metric on* $\mathfrak{g}$ *by* $\langle v^*, w^*\rangle = \langle Bv, w\rangle$. *This makes the decomposition* $\mathfrak{g} = V^* \oplus W$ *an orthogonal splitting and the isometry algebra of* M_B *is given by*
$$\mathfrak{isom}(M_B) = \mathfrak{so}_B(V) \oplus V^* \oplus W \tag{5}$$
with
$$\begin{aligned}\mathfrak{so}_B(V) &= \{A \in \mathfrak{so}(V)\,|\,[A, B] = 0\}\\ &= \{A \in \mathfrak{so}(V)\,|\,[A, v^*] = (Av)^*\}\,.\end{aligned} \tag{6}$$
- M_B *is indecomposable if and only if* B *is non-degenerate.*
- *Two Lorentzian spaces defined by symmetric maps* B_1 *and* B_2 *are isometric if and only if* B_1 *and* B_2 *are conformally equivalent.*

Definition 11. Let $\mathfrak{h} \subseteq \mathfrak{so}_B$. A *Clifford map* of $(M_B, \mathfrak{h})$ is a $\mathfrak{h} \oplus V^*$-equivariant linear map $\rho : \mathfrak{g} \oplus \mathfrak{h} \to \mathrm{C}\ell(W)$. Moreover, the image of ρ is considered to act on a Clifford module $S = S(W)$ and the restriction to $V^* \oplus \mathfrak{h}$ is fixed to be the spin-representation.

Definition 12. A *Clifford representation* of $(M_B, \mathfrak{h})$ is a Clifford map ρ that is a representation of $\mathfrak{g} \oplus \mathfrak{h}$ on the Clifford module S.

Notation 13. A Clifford map ρ is called *simple* if $\mathfrak{h} = 0$. It is called *irreducible* if S is, otherwise it is called *reducible*.

Proposition 14 ([30])**.** *Clifford maps correspond to spinor connections on* M_B *and Clifford representations correspond to flat spinor connections. The latter because of*
$$\mathcal{R}^\rho(x, y) = [\rho(x), \rho(y)] - \rho([x, y]_W) - \Gamma([x, y]_{V^*}) \tag{7}$$
for all $x, y \in W$.

Proposition 15. *A simple Clifford map of* M_B *is given by*
$$\rho(v^*) = \frac{1}{2}\Gamma_+ \hat{\otimes} Bv\,, \tag{8}$$
$$\rho(e_+) = \Gamma_+ \hat{\otimes} a\,, \tag{9}$$
$$\rho(e_-) = \sigma_- \hat{\otimes} c + \sigma_+ \hat{\otimes} d + \Gamma_+ \hat{\otimes} e - \Gamma_- \hat{\otimes} b\,, \tag{10}$$
$$\rho(w) = -\sigma_- \hat{\otimes} wb - \sigma_+ \hat{\otimes} \bar{b}w - \frac{1}{2}\Gamma_+ \hat{\otimes} s_{\bar{c},d}(w)\,, \tag{11}$$
with $a = \alpha + \beta\Gamma^*$ *and* $b = -\alpha + \beta\Gamma^*$ *if* $\dim(V)$ *is even and* $a = -b = \alpha$ *if* $\dim(V)$ *is odd.*

Theorem 16. *Any simple irreducible Clifford representation of M_B with $\rho(e_+) = 0$ is given by*

$$\begin{gathered}\rho(v^*) = \frac{1}{2}\Gamma_+ \hat{\otimes} Bv\,, \qquad \rho(w) = -\frac{1}{2}\Gamma_+ \hat{\otimes} s_{\bar{c},d}(w)\\ \rho(e_-) = \sigma_- \hat{\otimes} c + \sigma_+ \hat{\otimes} d + \Gamma_+ \hat{\otimes} e\end{gathered} \tag{12}$$

with $(\bar{c}, d)$ being a quadratic Clifford pair representing the symmetric map $-B$. In particular, $e \in \mathrm{C}\ell(V)$ is a free parameter of the representation.

Theorem 17. *A simple Clifford representation of M_B with $\rho(e_+) \neq 0$ is only possible in even dimension and for $B = -2\lambda\mathbb{1}$. It is given by*

$$\begin{aligned}\rho(v^*) &= -\lambda\Gamma_+ \hat{\otimes} v\,, \qquad \rho(e_+) = \alpha\Gamma_+ \hat{\otimes} \Pi^{\pm}\,,\\ \rho(e_-) &= \rho_0 \mathbb{1} \hat{\otimes} \mathbb{1} + (\alpha\Gamma_- + \beta\Gamma_+ + \sqrt{2(\alpha\beta+\lambda)}\sigma) \hat{\otimes} \Pi^{\mp}\\ &\quad + \sigma_- \hat{\otimes} c^{\pm}_{\mp} + \sigma_+ \hat{\otimes} d^{\mp}_{\pm} + \Gamma_+ \hat{\otimes} (e^+_- + e^-_+ + e^{\pm}_{\pm})\\ \rho(v) &= \alpha\sigma_- \hat{\otimes} \Pi^{\pm} v + \alpha\sigma_+ \hat{\otimes} \Pi^{\mp} v + \sqrt{\frac{\alpha\beta+\lambda}{2}}\Gamma_+ \hat{\otimes} v + \frac{1}{2}\Gamma_+ \hat{\otimes} s_{\bar{c}^{\pm}_{\mp}, d^{\mp}_{\pm}}(v)\,.\end{aligned} \tag{13}$$

The free parameters that describe the representation are the scalars α, β, ρ_0 and the Clifford element $e^{\pm}_{\pm}$. The further contributions are related by $\sqrt{2}\,\alpha s_{e^{\pm}_{\mp}, -e^{\mp}_{\pm}}(v) = \sqrt{\alpha\beta+\lambda}\, s_{\bar{c}^{\pm}_{\mp}, d^{\mp}_{\pm}}(v)$ for all $v \in V$.

Proposition 18. *(1) The simple Clifford representations of Theorem 16 extend to non-simple ones if and only if c, d and e are invariant with respect to $\mathfrak{h} \subset \mathfrak{so}_B(V)$.*

(2) The simple Clifford representations of Theorem 17 extend to non-simple ones for all choices of subalgebras of $\mathfrak{so}(V)$ for some special choices of parameters.

Definition 19. A *super extension* of $(M_B, \mathfrak{h})$ is a linear super space $\widehat{\mathfrak{g}}$ with

- (i) $\widehat{\mathfrak{g}}_0 = \mathfrak{g} \oplus \mathfrak{h}$,
- (ii) a Clifford map ρ of $(M_B, \mathfrak{h})$ that defines an invariant connection with curvature $\mathcal{R}^\rho$ on the Clifford module $S(W)$.
- (iii) $\widehat{\mathfrak{g}}_1 = \tilde{S}$ with $\tilde{S} \subseteq \ker(\mathcal{R}^\rho) \subset S(W)$ such that ρ acts on $\tilde{S}$.
- (iv) a symmetric map $\{\cdot,\cdot\} : \tilde{S} \otimes \tilde{S} \to \widehat{\mathfrak{g}}_0$ that is compatible with the Clifford map, i.e. $\{\rho(x)\xi, \eta\} + \{\xi, \rho(x)\eta\} = [x, \{\xi, \eta\}]$ for all $x \in \widehat{\mathfrak{g}}_0$, $\xi, \eta \in \tilde{S}$.

The product that is defined by $[x, y] := [x, y]$, $[x, \xi] = -[\xi, x] := \rho(x)\xi$ and $[\xi, \eta] := \{\xi, \eta\}$ for $x, y \in \widehat{\mathfrak{g}}_0$, $\xi, \eta \in \tilde{S}$ turns $\widehat{\mathfrak{g}}$ into an anti-commutative superalgebra. A super extension of $(M_B, \mathfrak{h})$ is called *supersymmetry extension* or *SUSY extension* if $\widehat{\mathfrak{g}}$ is a super Lie algebra, i.e. if

- (v) $[\{\xi, \xi\}, \xi] = 0$ for all $\xi \in \tilde{S}$

holds in addition to (i)-(iv).

Notation 20.
- A super extension is called *flat* if ρ is a Clifford representation, i.e. $\mathcal{R}^\rho = 0$.
- A super extension (SUSY extension) is called *restricted* if (iv) ((iv) or (v)) is only obtained for $\tilde{S}$ being a proper subset of $\ker(\mathcal{R}^\rho)$ subject to some further algebraic conditions. In particular, a non-restricted super extension with $\mathfrak{g}_1 = S$ is necessarily flat.
- A restriction is called *canonical restriction* if it commutes with the action of $\mathrm{C}\ell_{\mathrm{even}}(V)$.
- Moreover, the extension is called *minimal* if there is no subalgebra $\mathfrak{h}' \subset \mathfrak{h}$ such that the restriction to $\mathfrak{g} \oplus \mathfrak{h}'$ remains a super extension.
- In analogy to the notations introduced for Clifford maps, we call the super extension *irreducible* or *reducible* if ρ is irreducible or reducible, respectively. More precisely, if it is modeled on $S \otimes \mathbb{C}^N$ with S irreducible, we call it an N-super extension. Finally, the extension is called *simple* if $\mathfrak{h} = 0$
- The map sending $\tilde{S} \otimes \tilde{S}$ to zero satisfies all conditions of a SUSY extension. This extension is called *trivial* SUSY extension.

Definition 21. (1) Let C be a charge conjugation on S such that C_1 obeys $\Delta_1 = 1$ where C_1 and Δ_1 are defined by

$$\langle C_1(\xi, \eta), X\rangle := C(\xi, \Gamma(X)\eta) = \Delta_1 C(\eta, \Gamma(X)\xi) \tag{14}$$

for all $x \in W$. Then the projection of the bilinear map $\{\cdot,\cdot\} : S \otimes S \to \widehat{\mathfrak{g}}_0$ to W is defined by

$$\{\cdot,\cdot\}^W := C_1\,. \tag{15}$$

(2) If S is not irreducible but a twisted bundle, we may replace C by $C \otimes \tau$ with a bilinear form τ on the twisting factor. In this way we may also overcome the symmetry condition.

(3) The map $\{\cdot,\cdot\}^W$ is extended uniquely by two suitable maps $\{\cdot,\cdot\}^{V^*}$ and $\{\cdot,\cdot\}^{\mathfrak{so}}$ to an $V^* \oplus W$-equivariant map $\{\cdot,\cdot\} : S \otimes S \to \widehat{\mathfrak{g}}_0$.

Proposition 22. *Let $c, d \in \mathrm{C}\ell(V)$ be invariant with respect to $\mathfrak{so}_B(V)$. Then $\{\cdot,\cdot\}^{\mathfrak{so}}$ from Definition 21 takes its values in this subalgebra of $\mathfrak{so}(V)$ if and only if*

$$\begin{aligned} c &= c_0 + \hat{c}_0\Gamma^* \\ d &= d_0 + \hat{d}_0\Gamma^* \end{aligned} \tag{16}$$

if $B = \lambda\mathbb{1}$,

$$\begin{aligned} c &= (c_0 + \hat{c}_0\Gamma^*) + (c_1 + \hat{c}_1\Gamma^*)\Gamma_{I_1} \\ d &= (c_0 - (-1)^{\dim(V)}\hat{c}_0\Gamma^*) + (d_1 + \hat{d}_1\Gamma^*)\Gamma_{I_1} \end{aligned} \tag{17}$$

if the number of different eigenvalues of B is two, and

$$\begin{aligned} c &= (c_0 + \hat{c}_0\Gamma^*) + \sum_\alpha (c_\alpha + \hat{c}_\alpha\Gamma^*)\Gamma_{I_\alpha} \\ d &= (c_0 - (-1)^{\dim(V)}\hat{c}_0\Gamma^*) + \sum_\alpha (-1)^{\hat{I}_\alpha}(c_\alpha - (-1)^{\dim(V)}\hat{c}_\alpha\Gamma^*)\Gamma_{I_\alpha} \end{aligned} \tag{18}$$

if the number of different eigenvalues of B exceeds two. If the dimension of V is odd, we may choose $\hat{c}_\alpha = \hat{d}_\alpha = 0$ because $\Gamma^* = \pm\mathbb{1}$.

Theorem 23. *Let M_B a solvable Lorentzian symmetric space associated to the non-degenerate symmetric map B on V. Define the map*

$$\{\cdot,\cdot\} : S \otimes S \to V^* \oplus \mathfrak{so}(V) \oplus W \tag{19}$$

according to Definition 21. Then the Clifford representation of Theorem 16 yields a non-restricted irreducible super extension of M_B that is minimal with $\mathfrak{h}' = \mathfrak{so}_B(V)$ only if the $\mathrm{C}\ell(V)$-parameters obey $e = 0$, c, d leave the charge conjugation invariant, and (c, d) is

- *a pair of pseudo scalars if $B \sim \mathbb{1}$,*
- *a monomial or a pseudo-monomial quadratic Clifford pair if B admits two different eigenvalues, and*
- *a generalized monomial quadratic Clifford pair according to Propositions 7 and 8 if B admits more than two different eigenvalues.*

The first case is only possible if the dimension of M_B is divisible by four. In fact, the multiplicity of the eigenvalues is even for all up to at most two.

Theorem 24. *The Clifford representation (13) according to Theorem 17 gives rise to a super extension of M_B if and only if the space S is restricted and $\dim(M_B) = 0 \bmod 4$. The following cases are possible:*

(1) If S is restricted to $S' = \{\vec{\xi} \in S \,|\, \Gamma_+ \otimes \Pi^+\vec{\xi} = 0\}$ the Clifford representation has to be specified and the $V^* \oplus W$-invariant map $\{\cdot,\cdot\} : S' \otimes S' \to V^* \oplus W$ is given by

$$\{\vec{\xi}, \vec{\xi}\}^W = C_1(\vec{\xi}, \vec{\xi})\,,$$
$$\{\vec{\xi}, \vec{\xi}\}^{V^*} = \frac{1}{\lambda}\sum_\mu C(\vec{\xi}, (\sqrt{2\lambda}\sigma_+ - \alpha\Gamma_-)\hat{\otimes}\,\Gamma_\mu\Pi^-\vec{\xi})e^*_\mu\,.$$

(2) If S is restricted to $S'' = \{\vec{\xi} \in S \,|\, \Gamma_+ \otimes \Pi^+\vec{\xi} = \Gamma_- \otimes \Pi^-\vec{\xi} = 0\}$ the Clifford representation has to be specified.

(3) If S is restricted to $S''' = \{\vec{\xi} \in S \,|\, \Gamma_+\vec{\xi} = \Gamma_- \otimes \Pi^-\vec{\xi} = 0\}$ the Clifford representation has to be specified.

In either case the super extension is simple and in addition trivial in case (2) and (3).

For the sake of completeness, we note the components of $\{\cdot,\cdot\}$ *for the restriction in (1):*

$$\{\vec{\xi},\vec{\xi}\}_- = \pm 2\sqrt{2} i C^{(n)}(\xi_1^+,\xi_1^-)\,, \quad \{\vec{\xi},\vec{\xi}\}_+ = 0\,,$$
$$\{\vec{\xi},\vec{\xi}\}_\mu = \mp 2 i C^{(n)}(\xi_1^-,\Gamma_\mu \xi_2^-)\,,$$
$$\{\vec{\xi},\vec{\xi}\}_{-\mu} = \mp\sqrt{\tfrac{2}{\lambda}} i C^{(n)}(\xi_1^-,\Gamma_\mu\xi_2^-) + \tfrac{\sqrt{2}\alpha}{\lambda} i C^{(n)}(\xi_1^-,\Gamma_\mu\xi_1^-)\,.$$

Proposition 25. *Consider a super extension on* M_B *that is defined by a linear pair* (A,A)*. This yields a non-restricted SUSY extension in dimension four. In higher dimensions a non-trivial SUSY extension can be obtained only if the kernel of* A *is non trivial and the restriction is to a subset of* $\ker(\Gamma_+\otimes A)$*. In particular, this is possible in dimensions eight with no further restriction and in dimension ten if the restriction is either contained in* S_{10}^+ *or* S_{10}^-*.*

Theorem 26.

- *A super extension according to Theorem 24 (1) is a SUSY extension if and only if* $\alpha = 0$ *or*

$$\sum_\mu \left(C^{(n)}(\xi_2^-,\Gamma_\mu\xi_1^-)\Gamma_\mu\xi_1^- - C^{(n)}(\xi_1^-,\Gamma_\mu\xi_1^-)\Gamma_\mu\xi_2^-\right) = 0 \tag{20}$$

  for all $\vec{\xi} = (\xi_1^+,\xi_1^-,0,\xi_2^-)^t \in S'$*.*
- *In the remainig cases 2. and 3. we get a SUSY extension because the super extension is trivial.*

Proposition 27. *Irreducible super extensions of solvable Lorentzian symmetric spaces of dimension* $D \leq 12$ *are SUSY extensions that are at most canonically restricted if the symmetric map* B *and the quadratic Clifford pair* $(\bar{c},d)$ *are from Table 2 or 3.*[2,3,4]

The obstruction is given by

$$0 = C^{(n)}(\xi_2,\xi_2)d\xi_2 + \frac{1}{8}\sum_{\mu,\nu} C^{(n)}(\xi_2,(s_{d,\bar{c}}(e_\nu)\Gamma_\mu + \Gamma_\mu s_{\bar{c},d}(e_\nu))\xi_2)\Gamma_{\mu\nu}\xi_2 \tag{21}$$

and

$$\begin{aligned} 0 = {} & C^{(n)}(\xi_2,\xi_2)\bar{c}\xi_1 + \frac{1}{8}\sum_{\mu,\nu} C^{(n)}(\xi_2,(s_{d,\bar{c}}(e_\nu)\Gamma_\mu + \Gamma_\mu s_{\bar{c},d}(e_\nu))\xi_2)\Gamma_{\mu\nu}\xi_1 \\ & \pm \sum_\mu C^{(n)}(\xi_1,\Gamma_\mu\xi_2)s_{\bar{c},d}(e_\mu)\xi_2 \mp \sum_\mu C^{(n)}(\xi_1,s_{\bar{c},d}(e_\mu)\xi_2)\Gamma_\mu\xi_2 \end{aligned} \tag{22}$$

[2] Up to dimension twelve irreducible SUSY extensions are possible for $D = 4, 8, 9, 10, 11, 12$.

[3] We omit in the list the combinations and that don't yield SUSY extensions for any choice of parameters.

[4] The superscript ^{p} at the value $\hat{I}$ indicates a pseudo monomial pair.

where the sign depends on the choice of charge conjugation.

TABLE 2. Low dimensional, non-trivial, at most canonically restricted SUSY extensions with $\rho(e_+) = 0$

$n+2$	$\hat{I}$	Restriction	$(\bar{c}, d)$	B
4^-	2		$(-3\beta\Gamma_I, \beta\Gamma_I)$	$4\beta^2 \mathbb{1}_2$
8^+	1		$(-2\beta\Gamma_I, \beta\Gamma_I)$	$\beta^2 \begin{pmatrix} 9 & \\ & \mathbb{1}_5 \end{pmatrix}$
	5		$(2\beta\Gamma_I, \beta\Gamma_I)$	$\beta^2 \begin{pmatrix} 9 & \\ & \mathbb{1}_5 \end{pmatrix}$
	1^p		$(-2\beta(\cos(\phi) + \sin(\phi)\Gamma^*)\Gamma_I,$ $\beta(\cos(\phi) - \sin(\phi)\Gamma^*)\Gamma_I)$	$\beta^2 \cos(2\phi) \begin{pmatrix} 9 & \\ & \mathbb{1}_5 \end{pmatrix}$
	2	$\ker(\Gamma_+ \otimes \Pi^\pm)$	$(\beta\Gamma_I, 0)$	$\beta^2 \mathbb{1}_6$
9	1		$(-5\beta\Gamma_I, 3\beta\Gamma_I)$	$4\beta^2 \begin{pmatrix} 16 & \\ & \mathbb{1}_6 \end{pmatrix}$
	2		$(5\beta\Gamma_I, \beta\Gamma_I)$	$4\beta^2 \begin{pmatrix} 4\mathbb{1}_2 & \\ & 9\mathbb{1}_5 \end{pmatrix}$
10^+ 10^-	2	$S_{10}^\pm$	$(3\beta\Gamma_I, \beta\Gamma_I)$	$4\beta^2 \begin{pmatrix} 4\mathbb{1}_2 & \\ & \mathbb{1}_6 \end{pmatrix}$
	6	$S_{10}^\pm$	$(-3\beta\Gamma_I, \beta\Gamma_I)$	$4\beta^2 \begin{pmatrix} 4\mathbb{1}_2 & \\ & \mathbb{1}_6 \end{pmatrix}$
	2^p	$S_{10}^\pm$	(c, c), $c = \beta(\mp 2\mathbb{1} + \Gamma^*)\Gamma_I$	$4\beta^2 \begin{pmatrix} 4\mathbb{1}_2 & \\ & \mathbb{1}_6 \end{pmatrix}$
			$(c, -c)$, $c = \beta(\mathbb{1} \mp 2\Gamma^*)\Gamma_I$	$4\beta^2 \begin{pmatrix} 4\mathbb{1}_2 & \\ & \mathbb{1}_6 \end{pmatrix}$
11	3		$(-3\beta\Gamma_I, \beta\Gamma_I)$	$-4\beta^2 \begin{pmatrix} 4\mathbb{1}_3 & \\ & \mathbb{1}_6 \end{pmatrix}$

TABLE 3. Low dimensional, irreducible, at most canonically restricted SUSY extensions with $\rho(e_+) \neq 0$

$D = n + 2$	Restriction	B
4^-	$\ker(\Gamma_+ \otimes \Pi^\pm)$	$-2\lambda \mathbb{1}_4$
8^+	$\ker(\Gamma_+ \otimes \Pi^\pm)$	$-2\lambda \mathbb{1}_6$

Definition 28. An N-super extension is called *natural* if the bilinear form on S is defined by a twisting $C \otimes \tau$ and the Clifford map is defined by a pair $(c', d') = (c \otimes \tilde{\tau}, d \otimes \tilde{\tau})$ such that $\tilde{\tau}^2 \sim \mathbb{1}$, and τ as well as $\tau\tilde{\tau}$ are either symmetric or skew symmetric. We call this situation a $(\tau|\tilde{\tau})$-*twisting*.

Proposition 29. *Table 4 lists all possible natural 2-SUSY extensions in dimension 5,6, and 7. Therefore it fills some gaps for the missing dimensions in Table 2 by allowing a minimal extension by $N = 2$.*[5]

In this situation the obstruction is,

$$
\begin{aligned}
(\{\boldsymbol{\xi}, \boldsymbol{\xi}\}\boldsymbol{\xi})_o &= \sqrt{2}i \sum_{r,s,p=1}^{2} \sigma_i^{rs} \sigma_j^{op} C^{(n)}(\xi_{r;2}, \xi_{s;2}) \begin{pmatrix} \bar{c}\xi_{p;1} \\ d\xi_{p;2} \end{pmatrix} \\
&\pm \frac{i}{\sqrt{2}} \sum_{r,s,p=1}^{2} \sum_{\mu} \sigma_i^{rs} \sigma_j^{op} \big(C^{(n)}(\xi_{r;1}, \Gamma_\mu \xi_{s;2}) + C^{(n)}(\Gamma_\mu \xi_{r;2}, \xi_{s;1})\big) \begin{pmatrix} s_{\bar{c},d}(e_\mu)\xi_{p;2} \\ 0 \end{pmatrix} \\
&\mp \frac{i}{\sqrt{2}} \sum_{k=0}^{3} \sum_{r,s=1}^{2} \sum_{\mu} \eta_{ij}^k \sigma_k^{rs} \big(C^{(n)}(\xi_{r;1}, s_{\bar{c},d}(e_\mu)\xi_{s;2}) + \varepsilon_{ij}\varepsilon_j C^{(n)}(s_{\bar{c},d}(e_\mu)\xi_{r;2}, \xi_{s;1})\big)\big) \begin{pmatrix} \Gamma_\mu \xi_{o;2} \\ 0 \end{pmatrix} \\
&+ \frac{i}{4\sqrt{2}} \sum_{k=0}^{3} \sum_{r,s=1}^{2} \sum_{\mu,\nu} \eta_{ij}^k \sigma_k^{rs} C^{(n)}(\xi_{r;2}, (s_{d,\bar{c}}(e_\nu)\Gamma_\mu + \Gamma_\mu s_{\bar{c},d}(e_\nu))\xi_{s;2}) \begin{pmatrix} \Gamma_{\mu\nu}\xi_{o;1} \\ \Gamma_{\mu\nu}\xi_{o;2} \end{pmatrix}
\end{aligned}
$$

for $o = 1, 2$ with $\boldsymbol{\xi} = \begin{pmatrix} \vec{\xi}_1 \\ \vec{\xi}_2 \end{pmatrix}$ and $\vec{\xi}_p = \begin{pmatrix} \xi_{p;1} \\ \xi_{p;2} \end{pmatrix}$.

[5] Again, we omit in the list the combinations that don't yield SUSY extensions for any choice of parameters. Moreover, we omit the symmetry with respect to the chirality of the restrictions if present. Last but not least we omit those twistings for which the components of the algebra decouple to a sum of irreducible SUSY extensions.

Table 4. Low dimensional, at most canonically restricted, natural 2-SUSY extensions.

$n+2$	Twisting[†]	$\hat{I}$	Restriction	$(\bar{c}, d)$	B
5	$(\sigma_2\|\sigma_0)$	1		$(-3\beta\Gamma_I, \beta\Gamma_I)$	$4\beta^2\begin{pmatrix}4 & \\ & \mathbb{1}_2\end{pmatrix}$
	$(\sigma_2\|\tilde{\tau})$	0		$(3\beta\Gamma_I, \beta\Gamma_I)$	$-4\beta^2\mathbb{1}_3$
6^+ 6^-	$(\sigma_2\|\sigma_0)$	2	$S_6^\pm \otimes \mathbb{C}^2$	$(\beta\Gamma_I, 0)$	$\beta^2\mathbb{1}_4$
		2^p	$S_6^\pm \otimes \mathbb{C}^2$	$(\beta\Pi^\mp\Gamma_I, 0)$	$\beta^2\mathbb{1}_4$
	$(\sigma_2\|\tilde{\tau})$	0	$S_6^\pm \otimes \mathbb{C}^2$	$(2\beta\Gamma_I, \beta\Gamma_I)$	$-\beta^2\mathbb{1}_4$
		4	$S_6^\pm \otimes \mathbb{C}^2$	$(-2\beta\Gamma_I, \beta\Gamma_I)$	$-\beta^2\mathbb{1}_4$
		0^p	$S_6^\pm \otimes \mathbb{C}^2$	(c, c), $c = \beta(\mp 3\mathbb{1} + \Gamma^*)$	$-4\beta^2\mathbb{1}_4$
				$(c, -c)$, $c = \beta(\mathbb{1} \mp 3\Gamma^*)$	$-4\beta^2\mathbb{1}_4$

† $\tilde{\tau} \in \operatorname{span}\{\sigma_1, \sigma_2, \sigma_3\}$

Proposition 30. *In addition to Table 4 we list in Table 5 all possible natural 2-SUSY extensions up to dimension eleven.*

TABLE 5. Low dimensional, at most canonically restricted, natural 2-SUSY extensions

$n+2$	Twisting	$\hat{I}$	Restriction[‡]	$(\bar{c}, d)$	B
4^+	$(\sigma_2\|\sigma_3)$	0	$S_4^+ \oplus S_4^-$	$(3\beta\Gamma_I, \beta\Gamma_I)$	$-4\beta^2\mathbb{1}_2$
	$(\sigma_2\|\sigma_0)$	2	$S_4^+ \oplus S_4^-$	$(-3\beta\Gamma_I, \beta\Gamma_I)$	$4\beta^2\mathbb{1}_2$
4^-	$(\sigma_1\|\sigma_3)$	0	$S_4^+ \oplus S_4^-$	$(3\beta\Gamma_I, \beta\Gamma_I)$	$-4\beta^2\mathbb{1}_2$
	$(\sigma_1\|\sigma_0)$	2	$S_4^+ \oplus S_4^-$	$(-3\beta\Gamma_I, \beta\Gamma_I)$	$4\beta^2\mathbb{1}_2$
	$(\sigma_2\tilde{\tau}\|\tilde{\tau})^\flat$	1		$(\beta\Gamma_I, 0)$	$\beta^2\mathbb{1}_2$
		1^p		$(\beta(\cos(\phi)+\sin(\phi)\Gamma^*)\Gamma_I, 0)$	$\beta^2\cos(2\phi)\mathbb{1}_2$
8^- [8^+]	$(\sigma_2\|\tilde{\tau})^\dagger$ [$(\sigma_1\|\tilde{\tau})^\dagger$]	1	$S_8^+ \oplus S_8^-$	$(-2\beta\Gamma_I, \beta\Gamma_I)$	$\beta^2\begin{pmatrix}9 & \\ & \mathbb{1}_5\end{pmatrix}$
		5	$S_8^+ \oplus S_8^-$	$(2\beta\Gamma_I, \beta\Gamma_I)$	$\beta^2\begin{pmatrix}9 & \\ & \mathbb{1}_5\end{pmatrix}$
		1^p	$S_8^+ \oplus S_8^-$	$(-2\beta(\cos(\phi)+\sin(\phi)\Gamma^*)\Gamma_I,$ $\beta(\cos(\phi)-\sin(\phi)\Gamma^*)\Gamma_I)$	$\beta^2\cos(2\phi)\begin{pmatrix}9 & \\ & \mathbb{1}_5\end{pmatrix}$
10^+ 10^-	$(\sigma_2\tilde{\tau}\|\tilde{\tau})^\flat$	4	$S_{10}^+ \otimes \mathbb{C}^2$	$(\beta\Gamma_I, 0)$	$-\beta^2\mathbb{1}_8$
		4^p	$S_{10}^+ \otimes \mathbb{C}^2$	$(\beta\Pi^-\Gamma_I, 0)$	$-\beta^2\mathbb{1}_8$

$^\flat$ $\tilde{\tau} \in \mathrm{span}\{\sigma_1, \sigma_2, \sigma_3\}$

† $\tilde{\tau} \in \mathrm{span}\{\sigma_1, \sigma_2\}$

‡ $S_{4k}^+ \oplus S_{4k}^- \subset S_{4k} \oplus S_{4k} = S_{4k} \otimes \mathbb{C}^2$

Remark 31. Table 6 contains a list of some further SUSY extensions and is far from being complete.

Table 6. Some further SUSY extensions

$n+2$	Twisting	$\hat{I}$	Restr.	$(\bar{c}, d)$	B
$6\ (N=4)$	$(C^{(5)}\vert\tilde{\tau})^{\dagger}$	2	$S_6^+ \otimes S_5$	$(\beta\Gamma_I, 0)$	$\beta^2 \mathbb{1}_4$
		2^p	$S_6^+ \otimes S_5$	$(\beta\Pi^-\Gamma_I, 0)$	$\beta^2 \mathbb{1}_4$
$4^+\ (N=2)$ $[4^-\ (N=2)]$	$\tau = \sigma_2$ $[\tau = \sigma_1]$	0^p	$S_4^+ \oplus S_4^-$	$(c, c), c = \beta(2\mathbb{1} \otimes \sigma_3 + \Gamma^* \otimes \sigma_0)$	$-4\beta^2 \mathbb{1}_2$
				$(c, -c), c = \beta(\mathbb{1} \otimes \sigma_3 + 2\Gamma^* \otimes \sigma_0)$	$-4\beta^2 \mathbb{1}_2$

† $\tilde{\tau} \in \mathrm{C}\ell_1(\mathbb{R}^5)$

Bibliography

[1] Dimitry V. Alekseevsky and Vicente Cortés: Classification of N-(super)-extended Poincaré algebras and bilinear invariants of the spinor representation of $Spin(p,q)$. *Comm. Math. Phys.* **183** (1997), no. 3, 477-510.

[2] Dimitry V. Alekseevsky, Vicente Cortés, Chandrashekar Devchand, and Uwe Semmelmann: Killing Spinors are Killing vector fields in Riemannian supergeometry. *J. Geom. Phys.* **26** (1998), no. 1-2, 37-50.

[3] Natxo Alonso-Alberca, Ernesto Lozano-Tellechea, and Tomás Ortín: Geometric construction of Killing spinors and supersymmetry algebras in homogeneous spacetimes. *Class. Quant. Grav.* **19** (2002) no. 23, 6009-6024.

[4] Julius Wess and Jonathan Bagger: *Supersymmetry and supergravity.* (Princeton Series in Physics) Princeton University Press, 2. ed., 1992.

[5] Chrisitan Bär: Real Killing spinors and holonomy. *Comm. Math. Phys.* **154** (1993), no. 3, 509-521.

[6] Helga Baum, Thomas Friedrich, Ralf Grunewald, and Ines Kath: *Twistors and Killing spinors on Riemannian manifolds.* (Teubner Texts in Mathematics 124) B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1991.

[7] Helga Baum and Ines Kath: Parallel spinors and holonomy groups on pseudo-Riemannian spin manifolds. *Ann. Global Anal. Geom.* **17** (1999), no. 1, 1âĂŞ17.

[8] Helga Baum, Kordian Lärz, and Thomas Leistner: On the full holonomy group of manifolds with special holonomy. arXiv:1204.5657 [math.DG], 2012.

[9] Arthur L. Besse: *Einstein Manifolds.* (Classics in Mathematics) Springer Verlag, Reprint of the 1987 ed., 2008.

[10] Michel Cahen and Nolan Wallach: Lorentzian symmetric spaces. *Bull. Amer. Math. Soc.* **76** (1970), 585-591.

[11] Andreas Čap and Jan Slovák: *Parabolic Geometries I.* Mathematical Surveys and Monographs, 154. American Mathematical Society, 2009.

[12] Claude Chevalley: *The Algebraic Theory of Spinors and Clifford Algebras* (Collected Works, Vol. 2). Springer-Verlag, Berlin, 1997.

[13] Piotr T. Chruściel and Jerzy Kowalski-Glikman: The isometry group and Killing spinors for the pp wave space-time in $D = 11$ supergravity. *Phys. Lett. B* **149** (1984), no. 1-3, 107-110.

[14] Sidney Coleman and Jeffrey Mandula: All possible symmetries of the S matrix. *Phys. Rev.* **159** (1967), 1251-1256.

[15] Eugène Cremmer and Sergio Ferrara: Formulation of 11-dimensional supergravity in superspace. *Phys. Lett. B* **91** (1980), no. 1, 61-66.

[16] Leonard Eugene Dickson: On quaternions and their generalization and the history of the eight square theorem. *Ann. of Math. (2)* **20** (1919), 155-171.

[17] José Figueroa-O'Farrill: Lorentzian symmetric spaces in supergravity. arXiv:math/0702205v1 [math.DG], 2007.

[18] José Figueroa-O'Farrill: On parallelizable NS-NS backgrounds. . *Class. Quant. Grav.* **20** (2003), 3327-3340.

[19] José Figueroa-O'Farrill, Patrick Meessen, and Simon Philip: Supersymmetry and homogeneity of M-theory backgrounds. *Class. Quant. Grav.* **22** (2005), 207-226.

[20] José Figueroa-O'Farrill and George Papadopoulos: Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory. *J. High Energy Phys.* **8** (2001) Paper 36:26.

[21] José Figueroa-O'Farrill and Noel Hustler: Symmetric backgrounds of type IIB supergravity. Preprint, arXiv:1209.4844.

[22] S. James Gates, Jr., Marcus T. Grisaru, Martin Roček, and Warren Siegel: *Superspace or One thousand and one lessons in supersymmetry.* (Frontiers in Physics, 58) Benjamin/Cummings Publishing Co., 1983.

[23] Rudolf Haag, Jan T. Łopuszański, and Martin Sohnius: All possible generators of sypersymmetries of the S matrix. *Nuclear Phys. B* **88** (1975), 257-274.

[24] F. Reese Harvey: *Spinors and Calibrations.* (Perspectives in Mathematics) Academic Press Inc., 1990.

[25] Ines Kath and Martin Olbrich: On the structure of pseudo-Riemannian symmetric spaces. *Transform. Groups* **14** (2009), no. 4, 847-885.

[26] Dietrich Kramer, Hans Stephani, Eduard Herlt, and Malcolm MacCallum: *Exact solutions of Einstein's field equations* (Cambridge Monographs on Mathematical Physics). Cambridge University Press, 2. Ed, 2003.

[27] Frank Klinker: Supersymmetric Killing structures. *Comm. Math. Phys.* **255** (2005), no. 2, 419-467.

[28] Frank Klinker: The torsion of spinor connections and related structures. *SIGMA Symmetry Integrability Geom. Methods Appl.* **2** (2006), Paper 077, 28 pp.

[29] Frank Klinker: SUSY structures on deformed supermanifolds. *Differential Geom. Appl.* **26** (2008), no. 5, 566-582.

[30] Shoshichi Kobayashi and Katsumi Nomizu.: *Foundations of Differential Geometry, Vol. II.* Wiley Classics Library, John Wiley & Sons, Inc, 1996.

[31] Wolfgang Kühnel and Hans-Bert Rademacher: Conformal completion of $U(n)$-invariant Ricci-flat Kähler metrics at infinity. Dedicated to the memory of Paul Günther. *Z. Anal. Anwendungen* **16** (1997), no. 1, 113-117.

[32] Taichiro Kugo and Paul Townsend: Supersymmetry and the division algebras. *Nucl. Phys.* **B221** (1983), no. 2, 357-380.
[33] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn: *Spin geometry.* (Princeton Mathematical Series, 38) Princeton University Press, 1989.
[34] Thomas Leistner: On the classification of Lorentzian holonomy groups. *J. Differential Geom.* **76** (2007), no. 3, 423-484.
[35] Pertti Lounesto: *Clifford algebras and spinors.* (London Mathematical Society Lecture Note Series, 286) Cambridge University Press, 2. ed., 2001.
[36] Patrick Meessen: Small note on pp-wave vacua in 6 and 5 dimensions. *Phys. Rev. D* **65** (2002), 087501.
[37] Thomas Neukirchner: Solvable Pseudo-Riemannian Symmetric Spaces. arXiv:math/0301326 [math.DG], 2003.
[38] Joseph Putter: Maximal sets of anti-commuting skew-symmetric matrices. *J. London Math. Soc.* **42** (1967), 303-308.
[39] Andrea Santi: Superizations of Cahen-Wallach symmetric spaces and spin representations of the Heisenberg algebra. *J. Geom. Phys.* **60** (2010), 295-325.
[40] Hung-Hsi Wu: Holonomy groups of indefinite metrics. *Pacific J. Math.* **20** (1967), no. 2, 351-392.