QUADRATIC POISSON STRUCTURES IN DIMENSION FOUR

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ABSTRACT. We present here a complete list of quadratic Poisson structures in dimension four. For details on the decomposition of quadratic Poisson structures see [1].

1. The Tools

We shortly recall some results and notations taken over from [1].

We consider the volume form Ψ on an oriented n-dimensional manifold M. Ψ induces an isomorphism of poly-vector fields and differential forms $\Psi : \mathfrak{X}^k(M) \to \Omega^{n-k}(M)$. This isomorphism defines a derivation $D : \mathfrak{X}^k(M) \to \mathfrak{X}^{k-1}(M)$ by $D = \Psi^{-1} \circ d \circ \Psi$. We denote the set of polynomial poly-vector fields by $\mathfrak{P}^\ell \subset \mathfrak{X}^\ell(M)$. The set of polynomial poly-vector fields is bi-graded $\mathfrak{P}^\ell = \bigoplus_k \mathfrak{P}^{k,\ell}$ with

$$\mathfrak{P}^{k,\ell} = S^k(\Omega(M)) \otimes \Lambda^\ell(\mathfrak{X}(M)) .$$

$\mathfrak{P}^{k,\ell}$ decomposes with respect to $\mathfrak{sl}_n$ into two irreducible components $\mathfrak{P}^{k,\ell} = V_{k,\ell} \oplus V_{k-1,\ell-1}$. The derivation D is of degree $(-1-1)$. when restricted to $\mathfrak{P}^{k,\ell}$

Theorem 1.1 ([1])**.** *Every polynomial poly-vector field $A \in \mathfrak{P}^{(k,\ell)}$ admits a unique decomposition $A = A_0 + A_1$ with $A_0 \in V_{k,\ell} \subset \mathfrak{P}^{(k,\ell)}$ and $A_1 \in V_{k-1,\ell-1} \subset \mathfrak{P}^{(k,\ell)}$. Explicitly we have $DA = DA_1$, $DA_0 = 0$, $A_1 = DA \wedge e^{(k,\ell)}$, i.e.*

$$A = A_0 + DA \wedge e^{(k,\ell)} , \tag{1}$$

with $e^{(k,\ell)} = \dfrac{1}{n+k-\ell} x^m \partial_m$.

The Schouten bracket $[\cdot,\cdot] : \mathfrak{X}^k(M) \times \mathfrak{X}^\ell(M) \to \mathfrak{X}^{k+\ell-1}(M)$ yields the following Theorem.

Theorem 1.2 ([1])**.** *Let $A \in \mathfrak{P}^{(k,\ell)}$ and ℓ even with $A = A_0 + DA \wedge e^{(k,\ell)}$ cf. Theorem 1.1.*

(1) *Suppose $k \neq \ell$. Then $\big[A, A\big] = 0$ if and only if*

$$\big[A_0, A_0\big] = \frac{2(\ell-k)}{n+k-\ell} DA \wedge A_0 . \tag{2}$$

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(2) *For $k=\ell$ we have $\big[A,A\big]=0$ if and only if*

$$\big[A_0,A_0\big]=\big[DA,A_0\big]=0\,. \tag{3}$$

In dimension four we deal with quadratic bi-vectors and their trace given by linear vector fields. A Poisson structure Π is described by a pair (Π_θ, A) where Π_θ is a trace free Poisson structure and A is linear vector field that is compatible with Π_θ in the way that $L_A\Pi_\theta=[A,\Pi_\theta]=0$ holds, see Theorem 1.2. The linear vector field encodes the trace of Π and so is trace free, i.e. $A\in\mathfrak{sl}_4$. Furthermore, because Π_θ is trace free there exists a cubic tri-vector field L such that $\Pi_\theta=DL$. If we consider the isomorphism $\Psi:\mathfrak{X}^k(M)\to\Omega^{n-k}(M)$ we may write $L=\Psi^{-1}\theta$ with a cubic one-form θ. The trace free bi-vector is then given by $\Pi_\theta=\Psi^{-1}d\theta$. This explains the notation for the trace free part of Π. The compatibility condition may be reformulated by

$$\begin{aligned}[A,\Pi_\theta]&=D(A\wedge\Pi_\theta)=\Psi^{-1}\circ d\circ\Psi(A\wedge\Pi_\theta)=\Psi^{-1}\circ d\circ\imath_A\circ\imath_{\Pi_\theta}\Psi\\&=\Psi^{-1}\circ d\circ\imath_A d\theta=\Psi^{-1}(-\imath_A\circ d^2\theta+L_Ad\theta)=\Psi^{-1}\circ L_Ad\theta\\&=\Psi^{-1}\circ dL_A\theta=\Pi_{L_A\theta}\end{aligned} \tag{4}$$

and, therefore, reads as

(i) $L_A\theta=0\,.$

The Poisson condition on Π_θ translates to θ in the following way. The commutator of Π_θ with itself is a tri-vector field and via Ψ a one-form. We look at its components

$$(\Psi\circ[\Pi_\theta,\Pi_\theta])_m=(\Psi\circ D(\Pi_\theta\wedge\Pi_\theta))_m=\partial_m\Psi(\Pi_\theta\wedge\Pi_\theta)\,.$$

The function $\Psi(\Pi_\theta\wedge\Pi_\theta)$ that is the only coefficient in $d\theta\wedge d\theta$ is a homogeneous bi-quadratic polynomial. Its derivative vanishes if and only if it vanishes itself. So the Poisson condition on Π_θ can be written as

(ii) $d\theta\wedge d\theta=0.$

To characterize all quadratic Poisson structures in four dimensions we have to look for pairs of one-forms θ with cubic coefficients and trace free matrices A such that (i) and (ii) are fulfilled. We write $\Pi_{\theta,A}$ for the resulting Poisson structure. We call a one-form and a matrix compatible if they obey condition (i). This is a linear condition on the coefficients of the one-form. Because of Remark 1.3 below – which is an easy consequence of the compatibility of the wedge product and the Schouten bracket –, the compatibility of Ψ with linear diffeomorphisms, as well as (i) and (ii), we may restrict ourselves to matrices A that are in Jordan form. Condition (ii) is non-linear and yields algebraic relations of degree two for the coefficients.

Remark 1.3 ([1])**.** $\Pi_{\theta,A}$ and $\Pi_{\eta,B}$ are Poisson isomorphic if and only if there is a linear isomorphism L such that

$$\eta=\det L\,L^*\theta,\quad\text{and } B=LAL^{-1}\,.$$

For the explicit calculations in (i) we expand the cubic one-form in the form $\theta=\theta_kdx^k$ and

$$\theta_k=\sum_{0\le m\le n\le o\le 3}\theta_{k;mno}x^mx^nx^o \tag{5}$$

with $(x^0, x^1, x^2, x^3) := (t, x, y, z)$. For $A \in \mathfrak{sl}_4$ we have

$$L_A \theta_k = A^i{}_k \theta_i + A^i{}_j x^j \frac{\partial \theta_k}{\partial x^i} \tag{6}$$

and $L_A\theta = 0$ is a system on the 80 coefficients $\theta_{k;mno}$. We write the coefficients of the one-forms as vectors in the basis of the cubic polynomials as given in (5). In view of (6) and the restriction to the Jordan form of the matrix A we only need the following vectors:

$$\big(\theta_k,\ t\partial_t\theta_k,\ x\partial_x\theta_k,\ y\partial_y\theta_k,\ z\partial_z\theta_k,\ t\partial_x\theta_k,\ x\partial_y\theta_k,\ y\partial_z\theta_k,\ x\partial_t\theta_k,\ z\partial_y\theta_k\big) =$$

$$\begin{pmatrix}
\theta_{k;012} & \theta_{k;012} & \theta_{k;012} & \theta_{k;012} & 0 & 2\theta_{k;112} & 2\theta_{k;220} & \theta_{k;013} & 2\theta_{k;002} & 0 \\
\theta_{k;013} & \theta_{k;013} & \theta_{k;013} & 0 & \theta_{k;013} & 2\theta_{k;113} & \theta_{k;023} & 0 & 2\theta_{k;003} & \theta_{k;012} \\
\theta_{k;023} & \theta_{k;023} & 0 & \theta_{k;023} & \theta_{k;023} & \theta_{k;123} & 0 & 2\theta_{k;330} & 0 & 2\theta_{k;220} \\
\theta_{k;123} & 0 & \theta_{k;123} & \theta_{k;123} & \theta_{k;123} & 0 & 2\theta_{k;223} & 2\theta_{k;331} & \theta_{k;023} & 2\theta_{k;221} \\
\theta_{k;001} & 2\theta_{k;001} & \theta_{k;001} & 0 & 0 & 2\theta_{k;110} & \theta_{k;002} & 0 & 3\theta_{k;000} & 0 \\
\theta_{k;002} & 2\theta_{k;002} & 0 & \theta_{k;002} & 0 & \theta_{k;012} & 0 & \theta_{k;003} & 0 & 0 \\
\theta_{k;003} & 2\theta_{k;003} & 0 & 0 & \theta_{k;003} & \theta_{k;013} & 0 & 0 & 0 & \theta_{k;002} \\
\theta_{k;110} & \theta_{k;110} & 2\theta_{k;110} & 0 & 0 & 3\theta_{k;111} & \theta_{k;012} & 0 & 2\theta_{k;001} & 0 \\
\theta_{k;112} & 0 & 2\theta_{k;112} & \theta_{k;112} & 0 & 0 & 2\theta_{k;221} & \theta_{k;113} & \theta_{k;012} & 0 \\
\theta_{k;113} & 0 & 2\theta_{k;113} & 0 & \theta_{k;113} & 0 & \theta_{k;123} & 0 & \theta_{k;013} & \theta_{k;112} \\
\theta_{k;220} & \theta_{k;220} & 0 & 2\theta_{k;220} & 0 & \theta_{k;221} & 0 & \theta_{k;023} & 0 & 0 \\
\theta_{k;221} & 0 & \theta_{k;221} & 2\theta_{k;221} & 0 & 0 & 3\theta_{k;222} & \theta_{k;123} & \theta_{k;220} & 0 \\
\theta_{k;223} & 0 & 0 & 2\theta_{k;223} & \theta_{k;223} & 0 & 0 & 2\theta_{k;332} & 0 & 3\theta_{k;222} \\
\theta_{k;330} & \theta_{k;330} & 0 & 0 & 2\theta_{k;330} & \theta_{k;331} & 0 & 0 & 0 & \theta_{k;023} \\
\theta_{k;331} & 0 & \theta_{k;331} & 0 & 2\theta_{k;331} & 0 & \theta_{k;332} & 0 & \theta_{k;330} & \theta_{k;123} \\
\theta_{k;332} & 0 & 0 & \theta_{k;332} & 2\theta_{k;332} & 0 & 0 & 3\theta_{k;333} & 0 & 2\theta_{k;223} \\
\theta_{k;000} & 3\theta_{k;000} & 0 & 0 & 0 & \theta_{k;001} & 0 & 0 & 0 & 0 \\
\theta_{k;111} & 0 & 3\theta_{k;111} & 0 & 0 & 0 & \theta_{k;112} & 0 & \theta_{k;110} & 0 \\
\theta_{k;222} & 0 & 0 & 3\theta_{k;222} & 0 & 0 & 0 & \theta_{k;223} & 0 & 0 \\
\theta_{k;333} & 0 & 0 & 0 & 3\theta_{k;333} & 0 & 0 & 0 & 0 & \theta_{k;332}
\end{pmatrix}$$

2. The List

We use the notations developed above and examine the different possibilities of the Jordan form of the matrix A. We list the compatible one-forms, i.e. the one-forms that obey (i). In most cases we furthermore give the algebraic relation for the coefficients, see condition (ii). For special solutions we write down the Poisson structure associated to the one-form. After all we have to distinguish 43 different cases.

[A]: $A = \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix}$ with $a + b + c + d = 0$:

[A.1]: a, b, c, d nonzero

[A.1.1]: $|a|, |b|, |c|, |d|$ distinct and nonzero

We have to consider two subcases

[A.1.1.1]: $|a|, |b|, |c|, |d|$ distinct, nonzero, and no further coefficient in $L_A\theta$ vanishes.

In this case the only one-form compatible with A is

$$\Theta_A = \vartheta_0 xyz\,dt + \vartheta_1 tyz\,dx + \vartheta_2 txz\,dy + \vartheta_3 txy\,dz \tag{7}$$

This one-form also obeys $d\Theta_{\mathbf{A}} \wedge d\Theta_{\mathbf{A}} = 0$ such that $\Pi_{\mathbf{A}} := \Pi_{\Theta_{\mathbf{A}}}$ is a quadratic Poisson structure, explicitly

$$\boxed{\begin{aligned}\Pi_{\mathbf{A}} = \ & \vartheta_{23}tx\,\partial_t \wedge \partial_x + \vartheta_{31}ty\,\partial_t \wedge \partial_y + \vartheta_{12}tz\,\partial_t \wedge \partial_z \\ & + \vartheta_{03}xy\,\partial_x \wedge \partial_y + \vartheta_{20}xz\,\partial_x \wedge \partial_z + \vartheta_{01}yz\,\partial_y \wedge \partial_z\end{aligned}}$$

with $\vartheta_{ij} := \vartheta_j - \vartheta_i$. In particular, only the coefficients ${\Pi_{ij}}^{ij}$ are present.

[A.1.1.2]: $|a|, |b|, |c|, |d|$ distinct, nonzero, and $b = -3a$.

This is the exceptional case where at least one more coefficient in $L_A\theta$ vanishes although all entries in A have different absolute values. This coefficient is of the type $(3a + b)$. Because the problem is symmetric in a, b, c, d this is the only possibility. We write $A = a \cdot \mathrm{diag}(1, -3, k, 2 - k)$. To stay in the case of nonzero eigenvalues of distinct absolute values we have to exclude $k = \pm 1, \pm 3, 5, 0, 2$.

We have to consider two subcases.

[A.1.1.2.a]: $k \neq -\frac{1}{3}, \frac{7}{3}, -7, 9$

In this subcase the one-form compatible with A is given by

$$\theta = \Theta_{\mathbf{A}} + t^2(\alpha x\,dt + \hat{\alpha} t\,dx) \tag{8}$$

With $d\theta = d\Theta_{\mathbf{A}} + (3\hat{\alpha} - \alpha)t^2 dt \wedge dx$ condition (ii) yields

$$(3\hat{\alpha} - \alpha)\vartheta_{23} = 0.$$

and the two possibilities are

- $3\hat{\alpha} = \alpha$ and $\boxed{\Pi_\theta = \Pi_{\mathbf{A}}}$.
- $\vartheta_{23} = 0$, i.e. $\vartheta_2 = \vartheta_3$ and

$$\boxed{\begin{aligned}\Pi_\theta = \ & \vartheta_{21}t(y\,\partial_t \wedge \partial_y - z\,\partial_t \wedge \partial_z) + \vartheta_{02}x(y\,\partial_x \wedge \partial_y - z\,\partial_x \wedge \partial_z) \\ & + (\underline{\alpha} t^2 + \vartheta_{01}yz)\,\partial_y \wedge \partial_z\end{aligned}}$$

with $\underline{\alpha} := 3\hat{\alpha} - \alpha$.

[A.1.1.2.b]: $k \in \{-\frac{1}{3}, \frac{7}{3}, -7, 9\}$

These cases are the same up to renaming the variables, so we choose $k = 9$. In this case the one-form compatible with $A = a \cdot \mathrm{diag}(1, -3, 9, -7)$ is given by

$$\theta = \Theta_{\mathbf{A}} + t^2(\alpha x\,dt + \hat{\alpha} t\,dx) + x^2(\beta y dx + \hat{\beta} x\,dy) \tag{9}$$

The quadratic relations that single out the Poisson structures are given by

$$(3\hat{\alpha} - \alpha)\vartheta_{23} = (3\hat{\beta} - \beta)\vartheta_{30} = 0\,. \tag{10}$$

We assume $\vartheta_{23} = \vartheta_{30} = 0$, i.e. $\vartheta_3 = \vartheta_2 = \vartheta_0$. With $\underline{\alpha} := 3\hat{\alpha} - \alpha$ and $\underline{\beta} := 3\hat{\beta} - \beta$ this yields the Poisson structure

$$\boxed{\Pi_\theta = \vartheta_{31} ty\, \partial_t \wedge \partial_y + (\underline{\beta} x^2 - \vartheta_{31} tz)\partial_t \wedge \partial_z + (\underline{\alpha} t^2 + \vartheta_{31} yz)\partial_y \wedge \partial_z \,.}$$

[A.1.2]: $a = b$ and $|a|, |c|, |d|$ distinct and nonzero

[A.1.2.1]: $a = b$ and $|a|, |c|, |d|$ distinct and nonzero and no further entries in $L_A\theta$ vanish.

The one-form compatible with A is given by

$$\theta = \Theta_{\mathbf{A}} + yz(\alpha t\, dt + \hat{\alpha} x\, dx) + t^2(\beta z\, dy + \hat{\beta} y\, dz) + x^2(\gamma z\, dy + \hat{\gamma} y\, dz) \tag{11}$$

To yield a Poisson structure the coefficients have to satisfy the following equations

$$\begin{aligned} \beta(\hat{\alpha} - 2\hat{\gamma}) - \hat{\beta}(\hat{\alpha} - 2\gamma) - \alpha(\hat{\gamma} - \gamma) - 0 \\ \vartheta_3(\alpha - 2\beta) - \vartheta_2(\alpha - 2\hat{\beta}) + (\vartheta_0 + \vartheta_1)(\beta - \hat{\beta}) = 0 \\ \vartheta_3(\hat{\alpha} - 2\gamma) - \vartheta_2(\hat{\alpha} - 2\hat{\gamma}) - (\vartheta_0 + \vartheta_1)(\gamma - \hat{\gamma}) = 0 \end{aligned} \tag{12}$$

Two sample Poisson structures are:

a. $\beta = \hat{\beta}, \gamma = \hat{\gamma}$, and $\vartheta_{23} = 0$ with

$$\boxed{\begin{aligned} \Pi_\theta = \;& \vartheta_{01} yz\, \partial_y \wedge \partial_z - (\vartheta_{02} x - \eta t) z\, \partial_x \wedge \partial_z + (\vartheta_{02} x - \eta t) y\, \partial_x \wedge \partial_y \\ & + (\vartheta_{12} t - \xi x) z\, \partial_t \wedge \partial_z - (\vartheta_{12} t - \xi x) y\, \partial_t \wedge \partial_y \end{aligned}}$$

b. $\vartheta_2 = \vartheta_3 = 0$, $\vartheta_0 = -\vartheta_1$, $\gamma = \beta$, $\hat{\gamma} = \hat{\beta}$ and $\alpha = \hat{\alpha}$ with

$$\boxed{\begin{aligned} \Pi_\theta = \;& 2\vartheta_1 yz\, \partial_y \wedge \partial_z + (t^2 + x^2)(\hat{\beta} - \beta) \\ & -(\vartheta_1 x - \eta t) z\, \partial_x \wedge \partial_z + (\vartheta_1 x - \hat{\eta} t) y\, \partial_x \wedge \partial_y \\ & - (\vartheta_1 t + \eta x) z\, \partial_t \wedge \partial_z + (\vartheta_1 t + \hat{\eta} x) y\, \partial_t \wedge \partial_y \end{aligned}}$$

where $\xi := \hat{\alpha} - 2\gamma$, $\eta := \alpha - 2\beta$ and $\hat{\eta} := \alpha - 2\hat{\beta}$

[A.1.2.2]: $a = b$ and $|a|, |c|, |d|$ distinct and nonzero and $a = -3c$

As in case [A.1.1.2] the additional coefficient that may be zero is of the type $a + 3c$. The symmetry in the entries a and c (resp. d) is broken so that we also have to consider the case $3a + c$. But the vanishing of the latter would let the matrix be $A = a \cdot \mathrm{diag}(1, 1, -3, 1)$ which contradicts $a \neq d$. So we are really left with $a = -3c$ or[i] $A = c \cdot \mathrm{diag}(-3, -3, 1, 5)$ with compatible one-form

$$\begin{aligned} \theta = \Theta_{\mathbf{A}} + yz(\alpha t\, dt + \hat{\alpha} x\, dx) \\ + t^2(\beta z\, dy + \hat{\beta} y\, dz) + x^2(\gamma z\, dy + \hat{\gamma} y\, dz) \\ + y^2(\hat{\delta} y\, dt + \delta t\, dy) + y^2(\hat{\epsilon} y\, dx + \epsilon x\, dy) \end{aligned} \tag{13}$$

[i]This is $k = -3$ or 5 in [A.1.1.2].

In addition to (12) condition (ii) yields

$$\begin{aligned}(3\hat{\delta}-\delta)(2\hat{\gamma}-\hat{\alpha})+(3\hat{\epsilon}-\epsilon)\vartheta_{30}&=0\\(3\hat{\epsilon}-\epsilon)(2\hat{\beta}-\alpha)+(3\hat{\delta}-\delta)\vartheta_{31}&=0\end{aligned}\tag{14}$$

We get a new solution for $\hat{\beta}=\beta,\hat{\gamma}=\gamma,\vartheta_{32}=0$, and $\xi=-\vartheta_{21}$, $\eta=-\vartheta_{20}$ as well as $\underline{\epsilon}:=\epsilon-3\hat{\epsilon}=\delta-3\hat{\delta}=:\underline{\delta}$ (see [A.1.2.1]):

$$\boxed{\begin{aligned}\Pi_\theta=\ &\vartheta_{01}yz\,\partial_y\wedge\partial_z-(\vartheta_{02}(x-t)z+\underline{\delta}y^2)\partial_x\wedge\partial_z+\vartheta_{02}(x-t)y\,\partial_x\wedge\partial_y\\&+(\vartheta_{12}(t-x)z+\underline{\delta}y^2)\partial_t\wedge\partial_z-\vartheta_{12}(t-x)y\,\partial_t\wedge\partial_y\,.\end{aligned}}$$

[A.1.3]: $a=-b\neq0$, $c=-d\neq0$ and $|a|\neq|c|$

[A.1.3.1]: $a=-b\neq0$, $c=-d\neq0$, $|a|\neq|c|$ and no further entries in $L_A\theta$ vanish.

The one-form compatible with A is given by

$$\theta=\Theta_{\mathbf{A}}+xt(\alpha x\,dt+\hat{\alpha}t\,dx)+yz(\beta z\,dy+\hat{\beta}y\,dz)\tag{15}$$

and the relation that determines the Poisson structure is

$$\begin{aligned}(\beta-\hat{\beta})(\alpha-\hat{\alpha})&=0\\(\alpha-\hat{\alpha})\vartheta_{23}&=0\\(\beta-\hat{\beta})\vartheta_{01}&=0\end{aligned}\tag{16}$$

In this case we have two new solutions that are related by interchanging the connected pairs of variables $(t,x)\leftrightarrow(y,z)$. E.g. $\beta-\hat{\beta}=\vartheta_{23}=0$:

$$\boxed{\begin{aligned}\Pi_\theta=\ &\vartheta_{21}t(y\,\partial_t\wedge\partial_y-z\,\partial_t\wedge\partial_z)+\vartheta_{02}x(y\,\partial_x\wedge\partial_y-z\,\partial_x\wedge\partial_z)\\&+(\vartheta_{01}yz+\breve{\alpha}tx)\partial_y\wedge\partial_z\end{aligned}}$$

with $\breve{\alpha}=2(\alpha-\hat{\alpha})$.

[A.1.3.2]: $a=-b\neq0$, $c=-d\neq0$, $a=-3c$

As before the remaining possibilities are symmetric to this one. The matrix is[ii] $A=c\cdot\mathrm{diag}(-3,3,1,-1)$ and the compatible one-form

$$\begin{aligned}\theta=\Theta_{\mathbf{A}}&+xt(\alpha x\,dt+\hat{\alpha}t\,dx)+yz(\beta z\,dy+\hat{\beta}y\,dz)\\&+y^2(\gamma y\,dt+\hat{\gamma}t\,dy)+z^2(\delta z\,dx+\hat{\delta}x\,dz)\end{aligned}\tag{17}$$

There are contributions to (16), namely

$$(\hat{\gamma}-3\gamma)\vartheta_{31}=(\hat{\delta}-3\delta)\vartheta_{20}=(\hat{\delta}-3\delta)(\hat{\gamma}-3\gamma)=0$$

and new Poisson structures is given by

$\beta-\hat{\beta}=\vartheta_{23}=0$ and $\vartheta_{31}=\hat{\delta}-3\delta=0$:

$$\boxed{\Pi_\theta=(\vartheta_{30}xz+\underline{\gamma}y^2)\partial_x\wedge\partial_z+(\vartheta_{03}yz+\breve{\alpha}xt)\partial_y\wedge\partial_z+\vartheta_{03}xy\,\partial_x\wedge\partial_y}$$

with $\underline{\gamma}=3\gamma-\hat{\gamma}$, $\underline{\delta}=3\delta-\hat{\delta}$.

[ii]This is $k=-1$ or 3 in [A.1.1.2].

A similar solution is obtained by choosing $\beta - \hat{\beta} = \vartheta_{23} = 0$ and $\vartheta_{20} = \hat{\gamma} - 3\gamma = 0$.[iii]

[A.1.4]: $a = b = -c = -d \neq 0$, i.e. $A = a \cdot \mathrm{diag}(1, 1, -1, -1)$

The one-form compatible with A is given by

$$\begin{aligned}\theta = \Theta_{\mathbf{A}} &+ yz(\alpha_1 t\,dt + \hat{\alpha}_1 x\,dx)\\ &+ t^2(\beta_1 z\,dy + \hat{\beta}_1 y\,dz) + x^2(\gamma_1 z\,dy + \hat{\gamma}_1 y\,dz)\\ &+ tx(\alpha_2 y\,dy + \hat{\alpha}_2 z\,dz)\\ &+ y^2(\beta_2 x\,dt + \hat{\beta}_2 t\,dx) + z^2(\gamma_2 x\,dt + \hat{\gamma}_2 t\,dx)\\ &+ ty(\delta_1 y\,dt + \hat{\delta}_1 t\,dy) + tz(\delta_2 z\,dt + \hat{\delta}_2 t\,dz)\\ &+ xy(\delta_3 y\,dx + \hat{\delta}_3 x\,dy) + xz(\delta_4 z\,dx + \hat{\delta}_4 x\,dz)\end{aligned} \tag{18}$$

The Poisson structures are due to (ii) singled out by the following system on the coefficients:

$$\begin{aligned}0 &= 2(\delta_1 - \hat{\delta}_1)\vartheta_{13} + (\beta_1 - \hat{\beta}_1)(\beta_2 - \hat{\beta}_2) - (\alpha_1 - 2\hat{\beta}_1)(\alpha_2 - 2\hat{\beta}_2)\\ 0 &= 2(\delta_2 - \hat{\delta}_2)\vartheta_{21} + (\beta_1 - \hat{\beta}_1)(\gamma_2 - \hat{\gamma}_2) + (\alpha_1 - 2\beta_1)(\hat{\alpha}_2 - 2\hat{\gamma}_2)\\ 0 &= 2(\delta_3 - \hat{\delta}_3)\vartheta_{30} + (\gamma_1 - \hat{\gamma}_1)(\beta_2 - \hat{\beta}_2) + (\hat{\alpha}_1 - 2\hat{\gamma}_1)(\alpha_2 - 2\beta_2)\\ 0 &= 2(\delta_4 - \hat{\delta}_4)\vartheta_{02} + (\gamma_1 - \hat{\gamma}_1)(\gamma_2 - \hat{\gamma}_2) - (\hat{\alpha}_1 - 2\gamma_1)(\hat{\alpha}_2 - 2\gamma_2)\\ 0 &= (\beta_1 - \hat{\beta}_1)(\vartheta_1 + \vartheta_0) + (\alpha_1 - 2\hat{\beta}_1)\vartheta_2 - (\alpha_1 - 2\beta_1)\vartheta_3 + 2(\alpha_2 - 2\hat{\beta}_2)(\delta_2 - \hat{\delta}_2)\\ &\quad - 2(\hat{\alpha}_2 - 2\hat{\gamma}_2)(\delta_1 - \hat{\delta}_1)\\ 0 &= (\gamma_1 - \hat{\gamma}_1)(\vartheta_1 + \vartheta_0) - (\hat{\alpha}_1 - 2\hat{\gamma}_1)\vartheta_2 + (\hat{\alpha}_1 - 2\gamma_1)\vartheta_3 + 2(\hat{\alpha}_2 - 2\gamma_2)(\delta_3 - \hat{\delta}_3)\\ &\quad - 2(\alpha_2 - 2\beta_2)(\delta_4 - \hat{\delta}_4)\\ 0 &= (\beta_2 - \hat{\beta}_2)(\vartheta_3 + \vartheta_2) - (\alpha_2 - 2\hat{\beta}_2)\vartheta_0 + (\alpha_2 - 2\beta_2)\vartheta_1 + 2(\alpha_1 - 2\hat{\beta}_1)(\delta_3 - \hat{\delta}_3)\\ &\quad - 2(\hat{\alpha}_1 - 2\hat{\gamma}_1)(\delta_1 - \hat{\delta}_1)\\ 0 &= (\gamma_2 - \hat{\gamma}_2)(\vartheta_3 + \vartheta_2) - (\hat{\alpha}_2 - 2\hat{\gamma}_2)\vartheta_0 + (\hat{\alpha}_2 - 2\gamma_2)\vartheta_1 - 2(\hat{\alpha}_1 - 2\gamma_1)(\delta_2 - \hat{\delta}_2)\\ &\quad + 2(\alpha_1 - 2\beta_1)(\delta_4 - \hat{\delta}_4)\\ 0 &= 4(\delta_2 - \hat{\delta}_2)(\delta_3 - \hat{\delta}_3) - 4(\delta_1 - \hat{\delta}_1)(\delta_4 - \hat{\delta}_4) - (\hat{\alpha}_2 - 2\hat{\gamma}_2)(\alpha_2 - 2\beta_2)\\ &\quad - (\hat{\alpha}_1 - 2\hat{\gamma}_1)(\alpha_1 - 2\beta_1) + (\hat{\alpha}_2 - 2\gamma_2)(\alpha_2 - 2\hat{\beta}_2) + (\hat{\alpha}_1 - 2\gamma_1)(\alpha_1 - 2\hat{\beta}_1)\end{aligned} \tag{19}$$

[A.1.5]: $a = b = c = -\frac{1}{3}d \neq 0$, i.e.[iv] $A = a \cdot \mathrm{diag}(1, 1, 1, -3)$

[iii] There was a mistake in an earlier version that led to an additional wrong solution. Thanks to Prof. Hasan Gümral for pointing that out.

[iv] This is $k = 1$ in [A.1.1.2].

The one-form compatible with A is given by

$$\begin{aligned}\theta = \Theta_{\mathbf{A}} &+ yz(\alpha_1 t\,dt + \hat\alpha_1 x\,dx)\\ &+ t^2(\beta_1 z\,dy + \hat\beta_1 y\,dz) + x^2(\gamma_1 z\,dy + \hat\gamma_1 y\,dz)\\ &+ xz(\alpha_2 t\,dt + \hat\alpha_2 y\,dy)\\ &+ t^2(\beta_2 z\,dx + \hat\beta_2 x\,dz) + y^2(\gamma_2 z\,dx + \hat\gamma_2 x\,dz)\\ &+ tz(\alpha_3 x\,dx + \hat\alpha_3 y\,dy)\\ &+ x^2(\beta_3 z\,dt + \hat\beta_3 t\,dz) + y^2(\gamma_3 z\,dt + \hat\gamma_3 t\,dz)\\ &+ t^2(\delta_1 z\,dt + \hat\delta_1 t\,dz) + x^2(\delta_2 z\,dx + \hat\delta_2 x\,dz) + y^2(\delta_3 z\,dy + \hat\delta_3 y\,dz)\end{aligned} \tag{20}$$

[A.2]: a, b, c nonzero and $d = 0$

[A.2.1]: a, b, c distinct, nonzero and $d = 0$

We do not have to consider the absolute values of the entries of A because of the trace condition.

[A.2.1.1]: a, b, c distinct, nonzero, $d = 0$ and no further entry in $L_A\theta$ vanishes.

In this case the compatible one-form is

$$\theta = \Theta_{\mathbf{A}} + \hat\vartheta z^3\,dz \tag{21}$$

The Poisson structure is the same as in case [A.1.1.1], because $d(z^3\,dz) = 0$.

[A.2.1.2]: a, b, c distinct, nonzero, $d = 0$ and $b = -3a$,

This is up to symmetry the only possibility for additional nonzero entries in $L_A\theta$ and the only matrix obeying this condition is $A = a \cdot \mathrm{diag}(1, -3, 2, 0)$. The compatible one-form is

$$\theta = \Theta_{\mathbf{A}} + \hat\vartheta z^3\,dz + t^2(\alpha x\,dt + \hat\alpha t\,dx) \tag{22}$$

The Poisson structure is the same as in [A.1.1.2.a].

[A.2.2]: $a = b = -\frac{1}{2}c \neq 0 = d$

I.e. $A = a \cdot \mathrm{diag}(1, 1, -2, 0)$ with compatible one-form

$$\theta = \Theta_{\mathbf{A}} + \hat\vartheta z^3\,dz + yz(\alpha t\,dt + \hat\alpha x\,dx) + t^2(\beta z\,dy + \hat\beta y\,dz) + x^2(\gamma z\,dy + \hat\gamma y\,dz) \tag{23}$$

the Poisson structure is the same as in [A.1.2.1].

[A.3]: $a = -b \neq 0$ and $c = d = 0$

I.e. $A = a \cdot \mathrm{diag}(1, -1, 0, 0)$ with compatible one-form

$$\begin{aligned}\theta = \Theta_{\mathbf{A}} &+ \hat\vartheta_1 y^3\,dy + \hat\vartheta_2 z^3\,dz\\ &+ tx(\alpha y\,dy + \hat\alpha z\,dz) + y^2(\beta_1 x\,dt + \hat\beta_1 t\,dx) + z^2(\beta_2 x\,dt + \hat\beta_2 t\,dx)\\ &+ yz(\delta_1 z\,dy + \hat\delta_1 y\,dz) + xt(\delta_2 x\,dt + \hat\delta_2 t\,dx)\\ &+ y^2(\epsilon_1 z\,dy + \hat\epsilon_1 y\,dz) + z^2(\hat\epsilon_2 z\,dy + \epsilon_2 y\,dz)\end{aligned} \tag{24}$$

Condition (ii) is equivalent to

$$\begin{aligned}
2(\beta_1\hat{\beta}_2 - \hat{\beta}_1\beta_2) + \alpha(\beta_2 - \hat{\beta}_2) - \hat{\alpha}(\beta_1 - \hat{\beta}_1) &= 0\\
\vartheta_1(\alpha - 2\beta_1) - \vartheta_0(\alpha - 2\hat{\beta}_1) + (\vartheta_2 + \vartheta_3)(\beta_1 - \hat{\beta}_1) + (\delta_2 + \hat{\delta}_2)(\beta_1 - \hat{\beta}_1) &= 0\\
\vartheta_1(\alpha - 2\beta_2) - \vartheta_0(\alpha - 2\hat{\beta}_2) + (\vartheta_2 + \vartheta_3)(\beta_2 - \hat{\beta}_2) + (\delta_2 + \hat{\delta}_2)(\beta_2 - \hat{\beta}_2) &= 0\\
(\epsilon_1 - 3\hat{\epsilon}_1)(\vartheta_0 - \vartheta_1) + 2(\delta_1 - \hat{\delta}_1)(\beta_1 - \hat{\beta}_1) &= 0\\
(\epsilon_2 - 3\hat{\epsilon}_2)(\vartheta_0 - \vartheta_1) - 2(\delta_1 - \hat{\delta}_1)(\beta_2 - \hat{\beta}_2) &= 0\\
(\epsilon_1 - 3\hat{\epsilon}_1)(\beta_2 - \hat{\beta}_2) - (\epsilon_2 - 3\hat{\epsilon}_2)(\beta_1 - \hat{\beta}_1) + 2(\delta_1 - \hat{\delta}_1)(\vartheta_0 - \vartheta_1) &= 0\\
(\epsilon_1 - 3\hat{\epsilon}_1)(\beta_1 - \hat{\beta}_1) &= 0\\
(\epsilon_2 - 3\hat{\epsilon}_2)(\beta_2 - \hat{\beta}_2) &= 0
\end{aligned} \tag{25}$$

Two solution with associated Poisson structures are given by

a. $\beta_1 = \hat{\beta}_1$, $\beta_2 = \hat{\beta}_2$ and $\vartheta_1 = \vartheta_0$ with

$$\boxed{\begin{aligned}
\Pi_\theta = \ &\big((\vartheta_{23} + \check{\delta}_2)xt + 2\check{\delta}_1 yz - \underline{\epsilon_1}y^2 + \underline{\epsilon_2}z^2\big)\partial_t \wedge \partial_x\\
&+\big((\vartheta_{12} + \delta_2)z + \eta y\big)\big(t\partial_t \wedge \partial_z - x\partial_x \wedge \partial_z\big)\\
&+\big((\vartheta_{13} + \hat{\delta}_2)y + \xi z\big)\big(x\partial_x \wedge \partial_y - t\partial_t \wedge \partial_y\big)
\end{aligned}}$$

b. $\vartheta_2 = -\vartheta_3$, $\delta_2 = -\hat{\delta}_2$, $\alpha = \hat{\alpha}$, $\beta_1 = \beta_2$, $\hat{\beta}_1 = \hat{\beta}_2$ and $\vartheta_0 = \vartheta_1 = \check{\delta}_1 = \underline{\epsilon_1} = \underline{\epsilon_2} = 0$

$$\boxed{\begin{aligned}
\Pi_\theta = \ &\check{\beta}_1(y^2 + z^2)\partial_y \wedge \partial_z - 2(\vartheta_2 + \delta_2)xt\,\partial_t \wedge \partial_x\\
&+\big((\vartheta_2 + \delta_2)y + \hat{\eta}z\big)t\partial_t \wedge \partial_y + \big((\vartheta_2 + \delta_2)z + \hat{\eta}y\big)t\partial_t \wedge \partial_z\\
&-\big((\vartheta_2 + \delta_2)y + \eta z\big)x\partial_x \wedge \partial_y - \big((\vartheta_2 + \delta_2)z + \eta y\big)x\partial_x \wedge \partial_z
\end{aligned}}$$

with $\check{\beta}_1 := \hat{\beta}_1 - \beta_1$, $\check{\delta}_i := \hat{\delta}_i - \delta_i$, $\underline{\epsilon_i} := \epsilon_i - 3\hat{\epsilon}_i$, $\eta := \alpha - 2\beta_1$, $\xi := \hat{\alpha} - 2\beta_2$, and $\hat{\eta} := \alpha - 2\hat{\beta}_1$.

[A.4]: $a = b = c = d = 0$

The compatibility condition is empty in this case. This means that all one-forms are compatible with $A = 0$.

[B]: $A = \begin{pmatrix} a & 1 & & \\ & a & & \\ & & b & \\ & & & c \end{pmatrix}$ with $b + c = -2a$

This is the case of one Jordan block of size 2. The coefficients that belong to the variables t and x are coupled and the symmetry, which could be seen in [A], is broken.

[B.1]: a, b, c nonzero

[B.1.1]: a, b, c distinct and nonzero

[B.1.1.1]: a, b, c distinct, nonzero and $a \neq -3b$ and $a \neq -3c$

This is the case where no further coefficient that involve a, b, c in the Lie derivative vanishes. The compatible one-form is given by

$$\Theta_{\mathbf{B}} = \vartheta_0 tyz\, dt + \vartheta_1 yz(x\, dt - t\, dx) + t^2(\vartheta_2 z\, dy + \hat{\vartheta}_2 y\, dz) \tag{26}$$

The derivative of this one-form is

$$\begin{aligned} d\Theta_{\mathbf{B}} = {}& \vartheta_1 z(t\, dx \wedge dy - y\, dt \wedge dx - x\, dt \wedge dy) \\ & + \vartheta_1 y(t\, dx \wedge dz - z\, dt \wedge dx - x\, dt \wedge dz) \\ & + (\hat{\vartheta}_2 - \vartheta_2)t^2\, dy \wedge dz + (2\vartheta_2 - \vartheta_0)tz\, dt \wedge dy + (2\hat{\vartheta}_2 - \vartheta_0)ty\, dt \wedge dz \end{aligned}$$

obeys $d\Theta_{\mathbf{B}} \wedge d\Theta_{\mathbf{B}} = 0$, and the associated Poisson structure is

$$\boxed{\begin{aligned} \Pi_{\mathbf{B}} = {}& \vartheta_1 z(t\, \partial_t \wedge \partial_z - y\, \partial_y \wedge \partial_z + x\, \partial_x \wedge \partial_z) \\ & - \vartheta_1 y(t\, \partial_t \wedge \partial_y + z\, \partial_y \wedge \partial_z + x\, \partial_x \wedge \partial_y) \\ & + (\hat{\vartheta}_2 - \vartheta_2)t^2\, \partial_t \wedge \partial_x - (2\vartheta_2 - \vartheta_0)tz\, \partial_x \wedge \partial_z + (2\hat{\vartheta}_2 - \vartheta_0)ty\, \partial_x \wedge \partial_y \end{aligned}}$$

[B.1.1.2]: a, b, c distinct, nonzero and $a = -3b$

This is – up to symmetry – the only case where one more coefficient in $L_A\theta$ vanishes this gives two more solutions, namely

$$\theta = \Theta_{\mathbf{B}} + y^2(\alpha y\, dt + \hat{\alpha} t\, dy) \tag{27}$$

Condition (ii) translates into

$$(3\alpha - \hat{\alpha})\vartheta_1 = 0 \tag{28}$$

and we get a new Poisson structure, different from [B.1.1.1], for $\vartheta_1 = 0$. I.e.

$$\boxed{\begin{aligned} \Pi_\theta = {}& (\hat{\vartheta}_2 - \vartheta_2)t^2\, \partial_t \wedge \partial_x + (2\hat{\vartheta}_2 - \vartheta_0)ty\, \partial_x \wedge \partial_y \\ & - \big((2\vartheta_2 - \vartheta_0)tz + \underline{\alpha} y^2\big)\, \partial_x \wedge \partial_z \end{aligned}}$$

with $\underline{\alpha} = \hat{\alpha} - 3\alpha$.

Remark 2.1. Here and in what follows we always get new solutions by sending the coefficients to zero that comes with the coupled terms and we will discuss these cases in particular.

[B.1.2]: $a = -b = -c \neq 0$

The compatible one-form is

$$\theta = \Theta_{\mathbf{B}} + (\alpha y^2 + \hat{\alpha} z^2)(x\, dt - t\, dx) + yt(\beta y\, dt + \hat{\beta} t\, dy) + zt(\gamma z\, dt + \hat{\gamma} t\, dz) \tag{29}$$

We get the following equations from (ii)

$$\begin{aligned} \hat{\alpha}(\hat{\beta} - \beta) + \alpha(\gamma - \hat{\gamma}) &= 0 \\ 2\alpha(\hat{\vartheta}_2 + \vartheta_2 - \vartheta_0) - (\hat{\beta} - \beta)\vartheta_1 &= 0 \\ 2\hat{\alpha}(\hat{\vartheta}_2 + \vartheta_2 - \vartheta_0) - (\hat{\gamma} - \gamma)\vartheta_1 &= 0 \end{aligned} \tag{30}$$

with the two sample solutions

a. $\hat{\alpha} = \alpha = \vartheta_1 = 0$, i.e.

$$\boxed{\begin{aligned}\Pi_\theta =\ & (\hat\vartheta_2-\vartheta_2)t^2\,\partial_t\wedge\partial_x + (\check\gamma z + (2\hat\vartheta_2-\vartheta_0)y)t\,\partial_x\wedge\partial_y \\ & -(\check\beta y + (2\vartheta_2-\vartheta_0)z)t\,\partial_x\wedge\partial_z\end{aligned}}$$

with $\check\beta = 2(\hat\beta-\beta)$ and $\check\gamma = 2(\hat\gamma-\gamma)$.

b. $\hat\gamma=\gamma$, $\hat\beta=\beta$ and $\vartheta_2+\hat\vartheta_2-\vartheta_0=0$. (i.e. $2\hat\vartheta_2-\vartheta_0=\hat\vartheta_2-\vartheta_2$ and $2\vartheta_2-\vartheta_0=\vartheta_2-\hat\vartheta_2$), i.e.

$$\boxed{\begin{aligned}\Pi_\theta =\ & (2\alpha y+\vartheta_1 z)\big(t\,\partial_t\wedge\partial_z + x\,\partial_x\wedge\partial_z - y\,\partial_y\wedge\partial_z\big) \\ & -(2\hat\alpha z+\vartheta_1 y)\big(t\,\partial_t\wedge\partial_y + z\,\partial_y\wedge\partial_z + x\,\partial_x\wedge\partial_y\big) \\ & +(\hat\vartheta_2-\vartheta_2)t\big(z\,\partial_x\wedge\partial_z + y\,\partial_x\wedge\partial_y + t\partial_t\wedge\partial_x\big)\end{aligned}}$$

[B.1.3]: $a=b=-\frac{1}{3}c\neq 0$

The compatible one-form is

$$\begin{aligned}\theta = \Theta_{\mathbf{B}} + \alpha tz(x\,dt - t\,dx) + t^2(\beta z\,dt + \hat\beta t\,dz) + y^2(\gamma z\,dy + \hat\gamma y\,dz) \\ + y^2(\delta z\,dt + \hat\delta t\,dz) + \epsilon tyz\,dy \end{aligned} \tag{31}$$

The relations that single out the Poisson structures read

$$\begin{aligned} 3\alpha(\gamma-3\hat\gamma)+\vartheta_1(\delta+\epsilon-3\hat\delta) &= 0 \\ \alpha(\delta+\epsilon-3\hat\delta) &= 0 \\ \vartheta_1(\gamma-3\hat\gamma) &= 0 \\ \alpha\big((2\hat\vartheta_2-\vartheta_0)+(\hat\vartheta_2-\vartheta_2)\big)+\vartheta_1(\beta-3\hat\beta) &= 0 \end{aligned} \tag{32}$$

with the particular solutions

a. $\alpha=\vartheta_1=0$

$$\boxed{\begin{aligned}\Pi_\theta =\ & \big((\hat\vartheta_2-\vartheta_2)t^2 + (2\hat\delta-\epsilon)yt + \underline{\gamma}y^2\big)\,\partial_t\wedge\partial_x \\ & + \big((2\hat\vartheta_2-\vartheta_0)ty + (\hat\delta-\delta)y^2 + \underline{\beta}t^2\ \big)\,\partial_x\wedge\partial_y \\ & - \big((2\vartheta_2-\vartheta_0)tz - (2\delta-\epsilon)yz\big)\partial_x\wedge\partial_z\end{aligned}}$$

with $\underline{\beta}=\beta-3\hat\beta$ and $\underline{\gamma}=\gamma-3\hat\gamma$.

b. $\beta-3\hat\beta=\gamma-3\hat\gamma=0$, $\epsilon=2\delta=2\hat\delta$ and $2\hat\vartheta_2-\vartheta_0=-(\hat\vartheta_2-\vartheta_2)$ ($\Leftrightarrow 2\vartheta_2-\vartheta_0=-3(\hat\vartheta_2-\vartheta_2)$)

$$\boxed{\begin{aligned}\Pi_\theta = {} & \vartheta_1 z(t\,\partial_t\wedge\partial_z - y\,\partial_y\wedge\partial_z + x\,\partial_x\wedge\partial_z) \\ & -(\vartheta_1 y+\alpha t)(t\,\partial_t\wedge\partial_y + z\,\partial_y\wedge\partial_z + x\,\partial_x\wedge\partial_y) \\ & +(\hat\vartheta_2-\vartheta_2)t(t\,\partial_t\wedge\partial_x + 3t\,\partial_x\wedge\partial_z - t\,\partial_x\wedge\partial_y \\ & -2\alpha tz\,\partial_y\wedge\partial_z\end{aligned}}$$

[B.2]: $a = -\frac{1}{2}b \neq 0$ and $c = 0$

The compatible one-form is

$$\theta = \Theta_{\mathbf{B}} + \alpha z^3 dz \tag{33}$$

For the associated Poisson structure see [B.1.1.1] because $d(z^3\,dz) = 0$.

[B.3]: $a = 0$ and $c = -b \neq 0$

$$\theta = \Theta_{\mathbf{B}} + \alpha t^2(x\,dt - t\,dx) + \beta t^3\,dt + yz(\gamma z\,dy + \hat\gamma y\,dz) \tag{34}$$

This one-form induces a Poisson structure if and only if the following holds – cf. (ii)

$$\begin{aligned}\alpha(\hat\gamma-\gamma) &= 0 \\ \alpha(\hat\vartheta_2-\vartheta_2) &= 0 \\ \vartheta_1(\hat\gamma-\gamma) &= 0\end{aligned} \tag{35}$$

As before we have two special solutions

a. $\alpha = \vartheta_1 = 0$ with

$$\boxed{\begin{aligned}\Pi_\theta = {} & \big((\hat\vartheta_2-\vartheta_2)t^2 + \check\gamma yz\big)\,\partial_t\wedge\partial_x \\ & -(2\vartheta_2-\vartheta_0)tz\,\partial_x\wedge\partial_z + (2\hat\vartheta_2-\vartheta_0)ty\,\partial_x\wedge\partial_y\end{aligned}}$$

b. $\vartheta_2 = \hat\vartheta_2$ and $\check\gamma := 2(\hat\gamma-\gamma) = 0$

$$\boxed{\begin{aligned}\Pi_\theta = {} & \vartheta_1 z(t\,\partial_t\wedge\partial_z - y\,\partial_y\wedge\partial_z + x\,\partial_x\wedge\partial_z) \\ & -\vartheta_1 y(t\,\partial_t\wedge\partial_y + z\,\partial_y\wedge\partial_z + x\,\partial_x\wedge\partial_y) \\ & +(2\hat\vartheta_2-\vartheta_0)t\big(y\,\partial_x\wedge\partial_y - z\,\partial_x\wedge\partial_z\big) - 4\alpha t^2\partial_y\wedge\partial_z\end{aligned}}$$

[B.4]: $a = b = c = 0$

$$\begin{aligned}\theta = &(\alpha_1 t^2 + \alpha_2 ty + \alpha_3 tz + \alpha_4 y^2 + \alpha_5 yz + \alpha_6 z^2)(x\,dt - t\,dx)\\ &+ t^3(\beta_0\,dt + \gamma_0\,dy + \delta_0\,dz) + t^2 y(\beta_1\,dt + \gamma_1\,dy + \delta_1\,dz)\\ &+ t^2 z(\beta_2\,dt + \gamma_2\,dy + \delta_2\,dz) + ty^2(\beta_3\,dt + \gamma_3\,dy + \delta_3\,dz)\\ &+ tyz(\beta_4\,dt + \gamma_4\,dy + \delta_4\,dz) + tz^2(\beta_5\,dt + \gamma_5\,dy + \delta_5\,dz)\\ &+ y^3(\beta_6\,dt + \gamma_6\,dy + \delta_6\,dz) + y^2 z(\beta_7\,dt + \gamma_7\,dy + \delta_7\,dz)\\ &+ yz^2(\beta_8\,dt + \gamma_8\,dy + \delta_8\,dz) + z^3(\beta_9\,dt + \gamma_9\,dy + \delta_9\,dz)\end{aligned} \tag{36}$$

[C]: $A = \begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & -3a \end{pmatrix}$

This is the case of one Jordan block of size three.

[C.1]: $a \neq 0$

The coefficients that vanish non trivially are derived by evaluating

$$\big((L_A\theta_0)_{023}, (L_A\theta_0)_{113}, (L_A\theta_1)_{013}\big) = \\ = \big((L_A\theta_0)_{123}, (L_A\theta_1)_{023}, (L_A\theta_1)_{113}, (L_A\theta_2)_{013}\big) = 0\,.$$

The compatible one-form is

$$\theta = t^2(\alpha z\,dt + \hat{\alpha} t\,dz) + \beta tz(x\,dt - t\,dx) \\ + \delta tz(y\,dt - x\,dx + t\,dy) + (x^2 - 2ty)\big(\epsilon z\,dt + \hat{\epsilon} t\,dz\big) \tag{37}$$

and has to fulfill

$$\beta(3\hat{\epsilon} - \epsilon + \delta) = 0 \tag{38}$$

to produce a Poisson structure.

We have one solution ($\beta = 0$) where the term that couples x and t is missing. This is given by

$$\boxed{\begin{aligned}\Pi_\theta = &\ \big(\underline{\alpha} t^2 + (\Gamma_2 - 2\Gamma_1)ty + \tfrac{1}{2}(\Gamma_1 - \Gamma_2)x^2\big)\,\partial_x \wedge \partial_y\\ &- \Gamma_1(t^2\,\partial_t \wedge \partial_x + tx\,\partial_t \wedge \partial_y) - \Gamma_2(tz\,\partial_x \wedge \partial_z + xz\,\partial_y \wedge \partial_z)\end{aligned}}$$

with $\underline{\alpha} := 3\hat{\alpha} - \alpha$ and $\Gamma_1 = 2\hat{\epsilon} + \delta$ as well as $\Gamma_2 = 2\epsilon + \delta$

A second solution is given by $3\hat{\epsilon} - \epsilon + \delta = 0$ – which is the same as $3\Gamma_1 - \Gamma_2 = 0$. The associated Poisson structure is now

$$\boxed{\begin{aligned}\Pi_\theta = &\ \big(\underline{\alpha} t^2 - \beta tx\big)\,\partial_x \wedge \partial_y + \beta\big(3tz\,\partial_y \wedge \partial_z - t^2\partial_t \wedge \partial_y\big)\\ &+\Gamma_1\big((ty - x^2)\partial_x \wedge \partial_y - t^2\partial_t \wedge \partial_x - tx\partial_t \wedge \partial_y - 3tz\partial_x \wedge \partial_z - 3xz\partial_y \wedge \partial_z\big)\end{aligned}}$$

[C.2]: $a = 0$

The compatible one-form is

$$\begin{aligned}\theta = t^2(\alpha z\,dt + \hat{\alpha} t\,dz) + \beta(x^2 - 2ty)t\,dz \\ + \big(\gamma_1 tz + \gamma_2 t^2 + \gamma_3 z^2 + \gamma_4(x^2 - 2ty)\big)(x\,dt - t\,dx) \\ + \big(\delta_1 tz + \delta_2 t^2 + \delta_3 z^2 + \delta_4(x^2 - 2ty)\big)(y\,dt - x\,dx + t\,dy) \\ + (\epsilon_1 z + \epsilon_2 t)(x^2\,dt - 2tx\,dx + 2t^2\,dy) \\ + (\mu_1 t^3 + \mu_2 t^2 z + \mu_3 tz^2 + \mu_4 z^3)dt\end{aligned} \tag{39}$$

[D]: $A = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

The one-form compatible with this matrix is

$$\begin{aligned}\theta = \alpha t^2(x\,dt - t\,dx) + \beta_1 t(x^2\,dt - 2tx\,dx + t^2\,dy) + \beta_2 t^2(y\,dt - x\,dx + t\,dy) \\ + \gamma\big((3tz^2 - 3xyz + \tfrac{4}{3}y^3)dt + (3x^2 z - y^2 x - 3tyz)dx \\ + (4t^2 y - x^2 y - 3txz)dy + (x^3 - 3txy + 3t^2 z)dz\big) \\ + \delta_1\big((3x^2 z - 3tyz - y^2 x)dt \\ + (4y^2 - 3txz - x^2 y)dx + (x^3 + 3t^2 z - 3txy)dy\big) \\ + \delta_2(x^2 - 2ty)\big(z\,dt - y\,dx + x\,dy - t\,dz\big) \\ + \epsilon_1(x^3 - 3txy + 3t^2 z)dt \\ + \epsilon_2\big((4y^2 t - x^2 y - 3txz)dt + (x^3 + 3t^2 z - 3txy)dx\big) \\ + \epsilon_3(x^2 - 2ty)\big(x\,dt - t\,dx\big) \\ + \epsilon_4(x^2 - 2ty)\big(y\,dt - x\,dx + t\,dy\big) \\ + \epsilon_5 t^2(z\,dt - y\,dx + x\,dy + z\,dt)\end{aligned} \tag{40}$$

[E]: $A = \begin{pmatrix} a & 1 & & \\ & a & & \\ & & -a & 1 \\ & & & -a \end{pmatrix}$

[E.1]: $a \neq 0$

In this case the compatible one-form is

$$\begin{aligned}\theta = ty(\alpha y\,dt + \hat{\alpha} t\,dy) + ty\big(\beta(z\,dt - y\,dx) + \hat{\beta}(t\,dz - x\,dy)\big) \\ + (tz - xy)(\gamma y\,dt + \hat{\gamma} t\,dy) \\ + (tz - xy)\big(\delta(z\,dt - y\,dx) + \hat{\delta}(t\,dz - x\,dy)\big)\end{aligned} \tag{41}$$

Due to condition (ii) the Poisson structures must obey

$$\begin{aligned}\delta - \hat{\delta} = 0 \\ (\beta - \hat{\beta})\big((\beta - \hat{\beta}) + (\gamma + \hat{\gamma})\big) = 0\end{aligned} \tag{42}$$

$\delta = \hat{\delta}$ eliminates the derivative of the last summand in (42) so that it does not enter into the Poisson structure. A sample poisson structure is given by $\hat{\gamma} = \beta - \hat{\beta} - \gamma$:

$$\boxed{\begin{aligned} \Pi_\theta = \; & \breve{\alpha} ty \partial_x \wedge \partial_z \eta\big(t^2\, \partial_t \wedge \partial_x - y^2\, \partial_y \wedge \partial_z\big) \\ & + \xi\big(ty\, \partial_t \wedge \partial_z + tz\, \partial_x \wedge \partial_z\big) + \zeta\big(ty\, \partial_t \wedge \partial_z + xy\, \partial_x \wedge \partial_z\big) \end{aligned}}$$

with $\breve{\alpha} := 2(\alpha - \hat{\alpha})$, $\eta := \beta - \gamma$, $\xi := 3\beta - 2\hat{\beta} - \gamma$, and $\zeta := 2\hat{\beta} - \beta - \gamma$.

[E.2]: $a = 0$

The compatible one-form is

$$\begin{aligned} \theta = \; & ty(\alpha_1 y\, dt + \hat{\alpha}_1 t\, dy) + yt(\alpha_2 t\, dt + \hat{\alpha}_2 y\, dy) \\ & + y^3(\mu_1\, dt + \hat{\mu}_2\, dy) + t^3(\mu_2\, dt + \hat{\mu}_1\, dy) \\ & + ty\big(\beta_1(z\, dt - y\, dx) + \hat{\beta}_1(t\, dz - x\, dy)\big) \\ & + ty\big(\beta_2(z\, dy - y\, dz) + \hat{\beta}_2(x\, dt - t\, dx)\big) \\ & + y^2\big(\hat{\epsilon}_1(x\, dy - t\, dz) + \epsilon_2(z\, dt - y\, dx) + \hat{\epsilon}_3(z\, dy - y\, dz)\big) \\ & + t^2\big(\epsilon_1(z\, dt - y\, dx) + \hat{\epsilon}_2(x\, dy - t\, dz) + \epsilon_3(x\, dt - t\, dx)\big) \\ & + (tz - xy)(\gamma_1 y\, dt + \hat{\gamma}_1 t\, dy) \\ & + (tz - xy)\big(\delta_1(z\, dt - y\, dx) + \hat{\delta}_1(t\, dz - x\, dy)\big) \\ & + (tz - xy)\big(\delta_2(x\, dt - t\, dx) + \hat{\delta}_2(z\, dy - y\, dz)\big) \end{aligned} \tag{43}$$

[F]: $A = \begin{pmatrix} a & 1 & & \\ -1 & a & & \\ & & b & \\ & & & c \end{pmatrix}$ with $b + c = -2a$

This is the case of one pair of complex eigenvalues. Without loss of generality we may assume, that the imaginary part of this eigenvalue is equal to 1. We will see that in this case the symmetry in most cases is barely broken, i.e. we will not add as many solutions to the generic case [F.1.1.1] as we did before.

[F.1]: a, b, c nonzero

[F.1.1]: a, b, c distinct and nonzero

[F.1.1.1]: a, b, c distinct, nonzero, $a \neq -3b, -3c$

The compatible one-form is

$$\Theta_{\mathbf{F}} = yz\big(\vartheta_0(x\, dt - t\, dx) + \vartheta_1(t\, dt + x\, dx)\big) + (x^2 + t^2)\big(\vartheta_2 y\, dz + \hat{\vartheta}_2 z\, dy\big) \tag{44}$$

and obeys $d\Theta_{\mathbf{F}} \wedge d\Theta_{\mathbf{F}} = 0$ so that the associated Poisson structure is given by

$$\boxed{\begin{aligned} \Pi_{\mathbf{F}} = \; & (\vartheta_2 - \hat{\vartheta}_2)(t^2 + x^2)\, \partial_t \wedge \partial_x - 2\vartheta_0 yz\, \partial_y \wedge \partial_z \\ & -(\vartheta_0 t - \vartheta_1 x + 2\vartheta_2 x) y\, \partial_t \wedge \partial_y + (\vartheta_0 t - \vartheta_1 x + 2\hat{\vartheta}_2 x) z\, \partial_t \wedge \partial_z \\ & -(\vartheta_0 x + \vartheta_1 t - 2\vartheta_2 t) y\, \partial_x \wedge \partial_y + (\vartheta_0 x + \vartheta_1 t - 2\hat{\vartheta}_2 t) z\, \partial_x \wedge \partial_z \end{aligned}}$$

[F.1.1.2]: a, b, c distinct, nonzero, $a = -3b$

In the case of one complex eigenvalue with nonzero imaginary part there is no further contribution to the compatible one-form.

[F.1.2]: $a = b = -\frac{1}{3}c \neq 0$

The only extra contributions to the compatible one-form [F.1.1.1] will be in dy and dz:

$$\theta = \Theta_{\mathbf{F}} + y^2(\epsilon z\, dy + \hat{\epsilon} y\, dz). \tag{45}$$

Condition (ii) implies

$$(\epsilon - 3\hat{\epsilon})(\vartheta_2 - \hat{\vartheta}_2) = 0 \tag{46}$$

and a new Poisson structure is given by $\vartheta_2 = \hat{\vartheta}_2$:

$$\boxed{\begin{aligned}\Pi_\theta = \ & \underline{\epsilon} y^2 \partial_t \wedge \partial_x - 2\vartheta_0 yz\, \partial_y \wedge \partial_z \\ & + (\vartheta_0 t - \vartheta_1 x + 2\vartheta_2 x)\big(z\, \partial_t \wedge \partial_z - y\, \partial_t \wedge \partial_y\big) \\ & + (\vartheta_0 x + \vartheta_1 t - 2\vartheta_2 t)\big(z\, \partial_x \wedge \partial_z - y\, \partial_x \wedge \partial_y\big)\end{aligned}}$$

with $\underline{\epsilon} = 3\hat{\epsilon} - \epsilon$.

[F.1.3]: $a = -b = -c \neq 0$

The compatible one-form is

$$\begin{aligned}\theta = \Theta_{\mathbf{F}} & + (\alpha_1 y^2 + \alpha_2 z^2)(x\, dt - t\, dx) \\ & + (\hat{\alpha}_1 y^2 + \hat{\alpha}_2 z^2)(t\, dt + x\, dx) + (x^2 + t^2)(\beta_1 y\, dy + \beta_2 z\, dz)\end{aligned} \tag{47}$$

Inserting this into $d\theta \wedge d\theta = 0$ yields

$$\begin{aligned}\alpha_1(\hat{\alpha}_2 - \beta_2) - \alpha_2(\hat{\alpha}_1 - \beta_1) &= 0 \\ \alpha_1(\vartheta_2 + \hat{\vartheta}_2 - \vartheta_1) + \vartheta_0(\hat{\alpha}_1 - \beta_1) &= 0 \\ \alpha_2(\vartheta_2 + \hat{\vartheta}_2 - \vartheta_1) + \vartheta_0(\hat{\alpha}_2 - \beta_2) &= 0\end{aligned} \tag{48}$$

Two sample solutions are

a. $\vartheta_0 = \alpha_1 = \alpha_2 = 0$, i.e.

$$\boxed{\begin{aligned}\Pi_\theta = \ & (\vartheta_2 - \hat{\vartheta}_2)(t^2 + x^2)\, \partial_t \wedge \partial_x \\ & + \big((\vartheta_1 - 2\vartheta_2)y + 2(\hat{\alpha}_2 - \beta_2)z\big)\big(x\, \partial_t \wedge \partial_y - t\, \partial_x \wedge \partial_y\big) \\ & + \big((\vartheta_1 - 2\hat{\vartheta}_2)z + 2(\hat{\alpha}_1 - \beta_1)y\big)\big(t\, \partial_x \wedge \partial_z - x\, \partial_t \wedge \partial_z\big)\end{aligned}}$$

b. $\hat{\alpha}_1 = \beta_1$, $\hat{\alpha}_2 = \beta_2$ and $\vartheta_1 = \vartheta_2 + \hat{\vartheta}_2$, i.e.

$$\boxed{\begin{aligned}\Pi_\theta = \ & \tilde{\vartheta}(t^2 + x^2)\, \partial_t \wedge \partial_x - 2(\vartheta_0 yz + \alpha_1 y^2 + \alpha_2 z^2)\, \partial_y \wedge \partial_z \\ & - (\vartheta_0 yt + 2\alpha_2 zt + \tilde{\vartheta} xy)\, \partial_t \wedge \partial_y + (\vartheta_0 tz + 2\alpha_1 yt - \tilde{\vartheta} xz)\, \partial_t \wedge \partial_z \\ & - (\vartheta_0 xy + 2\alpha_2 zx - \tilde{\vartheta} ty)\, \partial_x \wedge \partial_y + (\vartheta_0 xz + 2\alpha_1 yx + \tilde{\vartheta} tz)\, \partial_x \wedge \partial_z\end{aligned}}$$

with $\tilde{\vartheta} = \vartheta_2 - \hat{\vartheta}_2$.

[F.2]: $a = -\frac{1}{2}b \neq 0$ and $c = 0$

The compatible one-form is

$$\theta = \Theta_{\mathbf{F}} + \vartheta_3 z^3\, dz \tag{49}$$

with Poisson structure $\Pi_{\mathbf{F}}$ because $d(z^3\, dz) = 0$.

[F.3]: $a = 0$ and $b = -c$

The compatible one-form is

$$\theta = \Theta_{\mathbf{F}} + zy(\gamma z\, dy + \hat{\gamma} y\, dz) + (x^2 + t^2)\big(\delta(x\, dt - t\, dx) + \hat{\delta}(t\, dt + x\, dx)\big) \tag{50}$$

The last term is a multiple of the derivative of $(t^2 + x^2)^2$ so that it does not enter the Poisson structure. the latter are singled out by the following equations

$$\begin{aligned} \delta(\hat{\gamma} - \gamma) &= 0 \\ \delta(\hat{\vartheta}_2 - \hat{\vartheta}_2) &= 0 \\ \vartheta_0(\hat{\gamma} - \gamma) &= 0 \end{aligned} \tag{51}$$

There are two new solutions, namely

a. $\delta = \vartheta_0 = 0$, which yields

$$\boxed{\begin{aligned} \Pi_\theta = \; & \big(\check{\gamma} yz + (\vartheta_2 - \hat{\vartheta}_2)(t^2 + x^2)\big)\, \partial_t \wedge \partial_x \\ & + (\vartheta_1 x - 2\vartheta_2 x) y\, \partial_t \wedge \partial_y - (\vartheta_1 x - 2\hat{\vartheta}_2 x) z\, \partial_t \wedge \partial_z \\ & - (\vartheta_1 t - 2\vartheta_2 t) y\, \partial_x \wedge \partial_y + (\vartheta_1 t - 2\hat{\vartheta}_2 t) z\, \partial_x \wedge \partial_z \end{aligned}}$$

and b. $\vartheta_2 = \hat{\vartheta}_2$, $\check{\gamma} := 2(\hat{\gamma} - \gamma) = 0$ with

$$\boxed{\begin{aligned} \Pi_\theta = \; & 2(2\delta(t^2 + x^2) - \vartheta_0 yz)\, \partial_y \wedge \partial_z \\ & - (\vartheta_0 t - \vartheta_1 x + 2\vartheta_2 x) y\, \partial_t \wedge \partial_y + (\vartheta_0 t - \vartheta_1 x + 2\hat{\vartheta}_2 x) z\, \partial_t \wedge \partial_z \\ & - (\vartheta_0 x + \vartheta_1 t - 2\vartheta_2 t) y\, \partial_x \wedge \partial_y + (\vartheta_0 x + \vartheta_1 t - 2\hat{\vartheta}_2 t) z\, \partial_x \wedge \partial_z \end{aligned}}$$

[F.4]: $a = b = c = 0$

The compatible one-form is

$$\begin{aligned} \theta = \Theta_{\mathbf{F}} & + (\alpha_1 y^2 + \alpha_2 z^2)\big(x\, dt - t\, dx\big) + (\hat{\alpha}_1 y^2 + \hat{\alpha}_2 z^2)\big(t\, dt + x\, dx\big) \\ & + (t^2 + x^2)\big(\beta y\, dy + \hat{\beta} z\, dz\big) + \vartheta_3 z^3\, dz + \hat{\vartheta}_3 y^3\, dy \\ & + (t^2 + x^2)\big(\delta(x\, dt - t\, dx) + \hat{\delta}(t\, dt + x\, dx)\big) + yz(\gamma z\, dy + \hat{\gamma} y\, dz) \\ & + y^2\big(\epsilon_1 z\, dy + \hat{\epsilon}_1 y\, dz\big) + z^2\big(\hat{\epsilon}_2 z\, dy + \epsilon_2 y\, dz\big) \end{aligned} \tag{52}$$

The system on the coefficients obtained by (ii) is

$$\begin{aligned}\delta\vartheta_{23} &= 0\\ \hat\alpha_1\vartheta_0+\alpha_1(\vartheta_3-\vartheta_1+\vartheta_2)+2\delta(\epsilon_1-3\hat\epsilon_1) &= 0\\ \hat\alpha_2\vartheta_0+\alpha_2(\vartheta_3-\vartheta_1+\vartheta_2)+2\delta(\epsilon_2-3\hat\epsilon_2) &= 0\\ 2(\gamma-\hat\gamma)\vartheta_0+\alpha_1(\epsilon_2-3\hat\epsilon_2)-\alpha_2(\epsilon_1-3\hat\epsilon_1) &= 0\\ \alpha_1(\epsilon_1-3\hat\epsilon_1) &= 0\\ \alpha_2(\epsilon_2-3\hat\epsilon_2) &= 0\\ (\alpha_1\hat\alpha_2-\hat\alpha_1\alpha_2)-2\delta(\gamma-\hat\gamma) &= 0\\ (\epsilon_1-3\hat\epsilon_1)\vartheta_0+2\alpha_1(\gamma-\hat\gamma) &= 0\\ (\epsilon_2-3\hat\epsilon_2)\vartheta_0-2\alpha_2(\gamma-\hat\gamma) &= 0\end{aligned} \tag{53}$$

and two Poisson structures are due to the following choices:

a. $\delta=\alpha_i=\vartheta_0=0$ and $\vartheta_1=\vartheta_2+\vartheta_3$

$$\boxed{\begin{aligned}\Pi_\theta = \;&\big(\underline{\epsilon_2}z^2-\underline{\epsilon_1}y^2-\vartheta_{23}(t^2+x^2)-\check\gamma yz\big)\partial_t\wedge\partial_x\\ &+\big(2\hat\alpha_1 y-\vartheta_{23}z\big)\big(t\partial_x\wedge\partial_z-x\partial_t\wedge\partial_z\big)\\ &+\big(2\hat\alpha_2 z-\vartheta_{23}y\big)\big(x\partial_t\wedge\partial_y-t\partial_x\wedge\partial_y\big)\end{aligned}}$$

b. $\vartheta_{23}=\hat\alpha_i=\underline{\epsilon_i}=\check\gamma=0$ and $\vartheta_1=2\vartheta_3$

$$\boxed{\begin{aligned}\Pi_\theta = \;&\big(-\alpha_1y^2-\alpha_2z^2-4\delta(t^2+x^2)-2\vartheta_0yz\big)\partial_y\wedge\partial_z\\ &+\big(2\alpha_1y+\vartheta_0z\big)\big(x\partial_x\wedge\partial_z+t\partial_t\wedge\partial_z\big)\\ &-\big(2\alpha_2z+\vartheta_0y\big)\big(t\partial_t\wedge\partial_y+x\partial_x\wedge\partial_y\big)\end{aligned}}$$

[G]: $A=\begin{pmatrix} a & d & & \\ -d & a & & \\ & & -a & 1\\ & & & -a\end{pmatrix}$ with $d\neq 0$

[G.1]: $a\neq 0$

The compatible one-form that obeys $d\Theta_{\mathbf{G}}\wedge d\Theta_{\mathbf{G}}=0$ is given by

$$\begin{aligned}\Theta_{\mathbf{G}} &= \tfrac12\vartheta_0y^2(x\,dt-t\,dx)+\tfrac12\vartheta_1y^2(t\,dt+x\,dx)\\ &\quad+\tfrac12\vartheta_2(t^2+x^2)(z\,dy-y\,dz)+\tfrac12\vartheta_3(t^2+x^2)y\,dy\end{aligned} \tag{54}$$

The Poisson structure determined by $\Theta_{\mathbf{G}}$ is

$$\boxed{\begin{aligned}\Pi_{\mathbf{G}} = \;&\vartheta_2xy\,\partial_t\wedge\partial_y-\vartheta_2ty\,\partial_x\wedge\partial_y-\vartheta_0y^2\,\partial_y\wedge\partial_z-\vartheta_3(t^2+x^2)\partial_t\wedge\partial_x\\ &+(\vartheta_0yt-\vartheta_1yx+\vartheta_2xz+\vartheta_3xy)\partial_t\wedge\partial_z\\ &+(\vartheta_0yx+\vartheta_1yt-\vartheta_2tz-\vartheta_3ty)\partial_x\wedge\partial_z\end{aligned}}$$

[G.2]: $a=0$

The compatible one-form in this case has four additional terms, of which two have vanishing derivative.

$$\theta = \Theta_{\mathbf{G}} + \tfrac{1}{4}(t^2+x^2)\big(\alpha(x\,dt - t\,dx) + \hat{\alpha}(t\,dt + x\,dx)\big) \\ + \tfrac{1}{4}\beta y^2(y\,dz - z\,dy) + \gamma y^3\,dy \quad (55)$$

Condition (ii) implies

$$\alpha\beta = \alpha\vartheta_3 = \beta\vartheta_0 = 0 \quad (56)$$

a. $\alpha = \vartheta_0 = 0$.

$$\boxed{\begin{aligned}\Pi_\theta =\ & \vartheta_2 xy\,\partial_t\wedge\partial_y - \vartheta_2 ty\,\partial_x\wedge\partial_y - (\vartheta_3(t^2+x^2) - \beta y^2)\partial_t\wedge\partial_x \\ & -(\vartheta_1 y - \vartheta_2 z - \vartheta_3 y)x\,\partial_t\wedge\partial_z + (\vartheta_1 y - \vartheta_2 z - \vartheta_3 y)t\,\partial_x\wedge\partial_z\end{aligned}}$$

b. $\beta = \vartheta_3 = 0$.

$$\boxed{\begin{aligned}\Pi_\theta =\ & \vartheta_2 xy\,\partial_t\wedge\partial_y - \vartheta_2 ty\,\partial_x\wedge\partial_y - (\vartheta_0 y^2 + \alpha(t^2+x^2))\,\partial_y\wedge\partial_z \\ & +(\vartheta_0 yt - \vartheta_1 yx + \vartheta_2 xz)\partial_t\wedge\partial_z + (\vartheta_0 yx + \vartheta_1 yt - \vartheta_2 tz)\partial_x\wedge\partial_z\end{aligned}}$$

[H]: $A = \begin{pmatrix} a & d & & \\ -d & a & & \\ & & -a & e \\ & & -e & -a \end{pmatrix}$ with $de \neq 0$

[H.1]: $a \neq 0$

[H.1.1]: $a \neq 0$ and $d^2 \neq e^2$

The compatible one-form that obeys $d\Theta_{\mathbf{H}} \wedge d\Theta_{\mathbf{H}} = 0$ is given by

$$\Theta_{\mathbf{H}} = \tfrac{1}{2}(y^2+z^2)\big(\vartheta_1(x\,dt - t\,dx) + \hat{\vartheta}_1(t\,dt + x\,dx)\big) \\ + \tfrac{1}{2}(t^2+x^2)\big(\vartheta_2(z\,dy - y\,dz) + \hat{\vartheta}_2(y\,dy + z\,dz)\big) \quad (57)$$

with Poisson structure

$$\boxed{\begin{aligned}\Pi_{\mathbf{H}} =\ & -\vartheta_2(t^2+x^2)\partial_t\wedge\partial_x - \vartheta_1(y^2+z^2)\partial_y\wedge\partial_z \\ & -\big((\hat{\vartheta}_1 - \hat{\vartheta}_2)xy - (\vartheta_1 ty + \vartheta_2 xz)\big)\partial_t\wedge\partial_z \\ & -\big((\hat{\vartheta}_1 - \hat{\vartheta}_2)tz + (\vartheta_1 xz + \vartheta_2 ty)\big)\partial_x\wedge\partial_y \\ & +\big((\hat{\vartheta}_1 - \hat{\vartheta}_2)xz - (\vartheta_1 tz - \vartheta_2 xy)\big)\partial_t\wedge\partial_y \\ & +\big((\hat{\vartheta}_1 - \hat{\vartheta}_2)ty + (\vartheta_1 xy - \vartheta_2 tz)\big)\partial_x\wedge\partial_z\end{aligned}}$$

[H.1.2]: $a \neq 0$ and $d^2 = e^2$

[H.1.2.1]: $a \neq 0$ and $d = e$

The compatible one-form is given by

$$\Theta_{\mathbf{H}+} = \Theta_{\mathbf{H}} + \tfrac{1}{2}(tz - xy)\big(\alpha_+(z\,dt - y\,dx) + \hat{\alpha}_+(x\,dy - t\,dz)\big) \\ + (xz + ty)\big(\beta_+(z\,dt - y\,dx) + \hat{\beta}_+(x\,dy - t\,dz)\big) \quad (58)$$

(ii) yields the system

$$\begin{aligned}(\alpha_+ \pm \hat\alpha_+)^2 - 2(\beta_+ \pm \hat\beta_+)(\vartheta_1 \pm \vartheta_2) + (\alpha_+ \pm \hat\alpha_+)(\hat\vartheta_1 - \hat\vartheta_2) = 0\\ 4(\beta_+ \pm \hat\beta_+)^2 + 2(\beta_+ \pm \hat\beta_+)(\vartheta_1 \pm \vartheta_2) - (\alpha_+ \pm \hat\alpha_+)(\hat\vartheta_1 - \hat\vartheta_2) = 0\\ 2(\alpha_+ \pm \hat\alpha_+)(\beta_+ \pm \hat\beta_+) + (\alpha_+ \pm \hat\alpha_+)(\vartheta_1 \pm \vartheta_2) + 2(\beta_+ \pm \hat\beta_+)(\hat\vartheta_1 - \hat\vartheta_2) = 0\end{aligned} \tag{59}$$

where we have to consider the upper sign. We find the sole solution

$$\alpha_+ + \hat\alpha_+ \;=\; \beta_+ + \hat\beta_+ \;=\; 0 \tag{60}$$

This shows that $\Pi_{\mathbf{H}}$ is not disturbed by the additional terms.

With these parameters the α-summand in (58) is proportional to $d\big((tz-xy)^2\big)$ so that this term does not contribute to the Poisson structure. The latter is

$$\begin{aligned}\Pi_{\mathbf{H}+} = \;&\Pi_{\mathbf{H}} + \beta_+\Big((t^2+x^2)\partial_t\wedge\partial_x - (y^2+z^2)\partial_y\wedge\partial_z\\ &+(ty-xz)\big(\partial_t\wedge\partial_z + \partial_x\wedge\partial_y\big) + (tz+xy)\big(\partial_x\wedge\partial_z - \partial_t\wedge\partial_y\big)\Big)\end{aligned}$$

[H.1.2.2]: $a \neq 0$ and $d = -e$

The compatible one-form is given by

$$\begin{aligned}\Theta_{\mathbf{H}-} = \Theta_{\mathbf{H}} + \tfrac{1}{2}(tz+xy)\big(\alpha_-(z\,dt + y\,dx) + \hat\alpha_-(x\,dy + t\,dz)\big)\\ + (xz-ty)\big(\beta_-(z\,dt + y\,dx) + \hat\beta_-(x\,dy + t\,dz)\big)\end{aligned} \tag{61}$$

Using the lower sign in system (59) gives

$$\alpha_- - \hat\alpha_- \;=\; \beta_- - \hat\beta_- \;=\; 0\,. \tag{62}$$

and β_- is the only new parameter in the Poisson structure:

$$\begin{aligned}\Pi_{\mathbf{H}-} = \;&\Pi_{\mathbf{H}} + \beta_-\Big((t^2+x^2)\partial_t\wedge\partial_x + (y^2+z^2)\partial_y\wedge\partial_z\\ &-(ty+xz)\big(\partial_t\wedge\partial_z - \partial_x\wedge\partial_y\big) - (xy-tz)\big(\partial_x\wedge\partial_z + \partial_t\wedge\partial_y\big)\Big)\end{aligned}$$

[H.2]: $a = 0$

[H.2.1]: $a = 0$, $d^2 \neq (5\pm 4)e^2$ and $e^2 \neq (5\pm 4)d^2$

The compatible one-form θ is given by

$$\begin{aligned}\theta = \Theta_{\mathbf{H}} + \Theta_0 := \Theta_{\mathbf{H}} + \tfrac{1}{4}(t^2+x^2)\big(\eta(x\,dt - t\,dx) + \hat\eta(t\,dt + x\,dx)\big)\\ + \tfrac{1}{4}(y^2+z^2)\big(\xi(z\,dy - y\,dz) + \hat\xi(y\,dy + z\,dz)\big)\end{aligned} \tag{63}$$

The $\,\hat{}\,$-terms are total derivatives and are not important for the Poisson structures. The latter are singled out by

$$\eta\xi \;=\; \eta\vartheta_2 \;=\; \xi\vartheta_1 \;=\; 0 \tag{64}$$

There are two new solutions– symmetric with respect to interchanging $(t,x) \leftrightarrow (y,z)$ –, e.g. $\xi = \vartheta_2 = 0$ with

$$\boxed{\begin{aligned}\Pi_\theta = \; & -(\eta(t^2+x^2) + \vartheta_1(t^2+x^2)\partial_y \wedge \partial_z \\ & + \big((\hat{\vartheta}_1 - \hat{\vartheta}_2)x - \vartheta_1 t\big)\big(z\partial_t \wedge \partial_y - y\partial_t \wedge \partial_z\big) \\ & + \big((\hat{\vartheta}_1 - \hat{\vartheta}_2)t + \vartheta_1 x\big)\big(y\partial_x \wedge \partial_z - z\partial_x \wedge \partial_y\big)\end{aligned}}$$

[H.2.2]: $a = 0$ and $d^2 = e^2$

[H.2.2.1]: $a = 0$ and $d = e$

The compatible one-form is given by

$$\begin{aligned}\theta = \; & \Theta_{\mathbf{H}+} + \Theta_0 \\ & + (t^2+x^2)\big(\delta_1(z\,dt - y\,dx) + \delta_2(y\,dt + z\,dx)\big) \\ & + \big(\epsilon_1(xz+ty) + \epsilon_2(xy - tz)\big)(x\,dt - t\,dx) \\ & + (y^2+z^2)\big(\hat{\delta}_1(x\,dy - t\,dz) + \hat{\delta}_2(t\,dy + x\,dz) \\ & + \big(\hat{\epsilon}_1(xz+ty) + \hat{\epsilon}_2(xt - yz)\big)(z\,dy - y\,dz) \\ & + (y^2+z^2)\big(\zeta_1(z\,dt - y\,dx) + \zeta_2(y\,dt + z\,dx)\big) \\ & + (t^2+x^2)\big(\hat{\zeta}_1(x\,dy - t\,dz) + \hat{\zeta}_2(t\,dy + x\,dz)\big)\end{aligned} \tag{65}$$

[H.2.2.2]: $a = 0$ and $d = -e$

The compatible one-form is given by

$$\begin{aligned}\theta = \; & \Theta_{\mathbf{H}-} + \Theta_0 \\ & + (t^2+x^2)\big(\delta_1(z\,dt + y\,dx) + \delta_2(y\,dt - z\,dx) \\ & + \big(\epsilon_1(xz-ty) + \epsilon_2(xy + tz)\big)(x\,dt - t\,dx) \\ & + (y^2+z^2)\big(\hat{\delta}_1(x\,dy + t\,dz) + \hat{\delta}_2(t\,dy - x\,dz) \\ & + \big(\hat{\epsilon}_1(xz-ty) + \hat{\epsilon}_2(xt + yz)\big)(z\,dy - y\,dz) \\ & + (y^2+z^2)\big(\zeta_1(z\,dt + y\,dx) + \zeta_2(y\,dt - z\,dx)\big) \\ & + (t^2+x^2)\big(\hat{\zeta}_1(x\,dy + t\,dz) + \hat{\zeta}_2(t\,dy - x\,dz)\big)\end{aligned} \tag{66}$$

[H.2.3]: $a = 0$ and $e^2 = 9d^2$

We obtain the case $d^2 = 9e^2$ by renaming the coordinates.

[H.2.3.1]: $a = 0$ and $e = 3d$

The compatible one-form is

$$\begin{aligned}\theta = \; & \Theta_{\mathbf{H}} + \Theta_0 \\ & + \alpha\big((tz - xy)(t\,dt - x\,dx) - (xz+ty)(x\,dt + t\,dx)\big) \\ & + \beta\big((tz - xy)(x\,dt + t\,dx) + (xz+ty)(t\,dt - x\,dx)\big) \\ & + \hat{\alpha}\big((x^2 - 3t^2)x\,dy + (t^2 - 3x^2)t\,dz\big) \\ & + \hat{\beta}\big((t^2 - 3x^2)t\,dy - (x^2 - 3t^2)x\,dz\big)\end{aligned} \tag{67}$$

[H.2.3.2]: $a = 0$ and $e = -3d$

The compatible one-form is

$$\begin{aligned}\theta = {}& \Theta_{\mathbf{H}} + \Theta_0 \\ &+ \alpha\big((tz+xy)(t\,dt - x\,dx) - (xz-ty)(x\,dt + t\,dx)\big) \\ &+ \beta\big((tz+xy)(x\,dt + t\,dx) + (xz-ty)(t\,dt - x\,dx)\big) \\ &+ \hat{\alpha}\big((x^2-3t^2)x\,dy - (t^2-3x^2)t\,dz\big) \\ &+ \hat{\beta}\big((t^2-3x^2)t\,dy + (x^2-3t^2)x\,dz\big)\end{aligned} \tag{68}$$

To get the Poisson structures from θ in the cases $d = \pm 3e$ the coefficients have to obey

$$\begin{aligned}\xi\vartheta_1 &= 0 \\ \eta\xi &= 0 \\ (\alpha \mp 3\hat{\alpha})^2 + (\beta \mp 3\hat{\beta})^2 \mp \vartheta_2\eta &= 0 \\ (\alpha \mp 3\hat{\alpha})(\vartheta_1 \pm \vartheta_2) - (\beta \mp 3\hat{\beta})(\hat{\vartheta}_2 - \hat{\vartheta}_1) &= 0 \\ (\beta \mp 3\hat{\beta})(\vartheta_1 \pm \vartheta_2) + (\alpha \mp 3\hat{\alpha})(\hat{\vartheta}_2 - \hat{\vartheta}_1) &= 0\end{aligned} \tag{69}$$

From the last two equations we get that $\underline{\beta} = \beta \mp 3\hat{\beta}$ and $\underline{\alpha} = \alpha \mp 3\hat{\alpha}$ can only be chosen non trivial if $\vartheta_1 \pm \vartheta_2 = \hat{\vartheta}_2 - \hat{\vartheta}_1 = 0$. We are left with [H.1.1] for $\eta = \xi = 0$ and [H.2.1] for $\vartheta_1 = \eta = 0$ or $\vartheta_2 = \xi = 0$, because of the third equation.

The only new Poisson structure is given by $\vartheta_1 \pm \vartheta_2 = \hat{\vartheta}_2 - \hat{\vartheta}_1 = \xi = 0$ and $\eta\vartheta_2 \gtrless 0$ with $\underline{\alpha}^2 + \underline{\beta}^2 = \pm\vartheta_2\eta$

$$\boxed{\begin{aligned}\Pi_\theta = {}& -\vartheta_2(t^2+x^2)\partial_t \wedge \partial_x - \big(\vartheta_1(y^2+z^2) + \eta(t^2+x^2)\big)\partial_y \wedge \partial_z \\ &+\big(\vartheta_1(ty \mp xz) + 2\underline{\beta}tx + \underline{\alpha}(t^2-x^2)\big)\partial_t \wedge \partial_z \\ &-\big(\vartheta_1(xz \mp ty) + 2\underline{\beta}tx + \underline{\alpha}(t^2-x^2)\big)\partial_x \wedge \partial_y \\ &-\big(\vartheta_1(tz \pm xy) + 2\underline{\alpha}tx - \underline{\beta}(t^2-x^2)\big)\partial_t \wedge \partial_y \\ &+\big(\vartheta_1(xy \pm tz) - 2\underline{\alpha}tx + \underline{\beta}(t^2-x^2)\big)\partial_x \wedge \partial_z\end{aligned}}$$

References

[1] Frank Klinker. Polynomial poly-vector fields. *Int. Electron. J. Geom.* **2** (2009), no.1, 55-73..