

THE SIGNS

SUPPLEMENT TO THE AUTHOR'S DISSERTATION SUPERSYMMETRIC KILLING STRUCTURES

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ABSTRACT. We use the notations introduced in [1, Sect. 1.2] to explain how we derive the signs cf. table 3 therein. The calculations given below are an extended version of the calculations for the Lorentzian case which have their origin in [2].

We denote by t the number of time like directions in the pseudo Riemannian metric of \mathbb{R}^D and by $\sigma = 2t - D$ its signature.

Consider the gamma-matrices γ_μ with $\gamma_\mu^2 = 1$ for $1 \leq \mu \leq t$ and $\gamma_\mu^2 = -1$ for $t + 1 \leq \mu \leq D$.

Furthermore we assume D to be even.

The different transformations are denoted by

$$A_\pm^\dagger \gamma_\mu A_\pm = \pm \gamma_\mu^* \tag{1}$$

$$B^\dagger \gamma^\mu B = (-)^{t+1} \gamma_\mu^\dagger \tag{2}$$

For B we have the explicit description

$$B = \gamma^1 \dots \gamma^t \tag{3}$$

with

$$B^\dagger = (-)^{\frac{1}{2}t(t-1)} B. \tag{4}$$

From (1) we get

$$(A_\pm^* A_\pm)^T \gamma_\mu (A_\pm A_\pm^*) = A_\pm^T A_\pm^\dagger \gamma_\mu A_\pm A_\pm^* = \pm A_\pm^T \gamma_\mu^* A_\pm^* = \pm (A_\pm^\dagger \gamma_\mu A_\pm)^* = \gamma_\mu$$

which yields $A_\pm A_\pm^* = \epsilon_\pm \mathbf{1}$ or equivalently

$$A_\pm^T = \epsilon_\pm A_\pm. \tag{5}$$

We define the charge conjugation by

$$C_\pm = A_\pm B^*. \tag{6}$$

These matrices satisfy

$$C_\pm^\dagger \gamma_\mu C_\pm = B^T A_\pm^\dagger \gamma_\mu A_\pm B^* = \pm B^T \gamma_\mu^* B^* = \pm (-)^{t+1} \gamma_\mu^T \tag{7}$$

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with

$$C_{\pm}^T = B^\dagger A_{\pm}^T \stackrel{(5)(4)}{=} (-)^{\frac{1}{2}t(t-1)} \epsilon_{\pm} B A_{\pm} \stackrel{(1)(3)(6)}{=} (\pm)^t (-)^{\frac{1}{2}t(t-1)} \epsilon_{\pm} C. \quad (8)$$

This yields the following symmetries for the **spin**-invariant morphisms

$$(C_{\pm}^\dagger \gamma_{\mu_1 \dots \mu_k})^T = (\pm)^t (-)^{\frac{1}{2}t(t-1)} (-)^{k(t+1)} (\pm)^k (-)^{\frac{1}{2}k(k-1)} \epsilon_{\pm} (C_{\pm}^\dagger \gamma_{\mu_1 \dots \mu_k}) \quad (9)$$

Compare with [1, Section 2.A, Lemma 2.8] where we called the prefactor Δ_{\pm}^k which has the periodicities $\Delta_{\pm}^{k+2} = -\Delta_{\pm}^k$ and therefore $\Delta_{\pm}^{k+4} = \Delta_{\pm}^k$.

Proposition 1. *The signs ϵ_{\pm} in (5) are given by [1, Table 3] which is*

σ	0	1	2	3	4	5	6	7
ϵ_+	+	+	+	/	-	-	-	/
ϵ_-	+	/	-	-	-	/	+	+

In the case of even dimension (i.e. even signature) we may explicitly write the sign as

$$\epsilon_{\pm} = -\sqrt{2} (\pm)^t (-)^{\frac{1}{2}t(t-1)} \cos \frac{(3 \pm (-)^{t+1} D) \pi}{4} \quad (10)$$

We count the skew symmetric and symmetric matrices among (9) and get the numbers

$$a_{\pm} = \frac{1}{2} \sum_{k=0}^D (1 - \Delta_{\pm}^k) \binom{D}{k} \quad (11)$$

$$s_{\pm} = \frac{1}{2} \sum_{k=0}^D (1 + \Delta_{\pm}^k) \binom{D}{k} \quad (12)$$

Furthermore we know that

$$2a_{\pm} = 2^{\frac{D}{2}} (2^{\frac{D}{2}} - 1) \quad \text{and} \quad 2s_{\pm} = 2^{\frac{D}{2}} (2^{\frac{D}{2}} + 1). \quad (13)$$

We use the abbreviation $\rho_{\pm} := (\pm)^t (-)^{\frac{1}{2}t(t-1)} \epsilon_{\pm}$ and the identities

$$(-)^{\frac{1}{2}k(k-1)} = \frac{1}{2} ((1-i)i^k + (1+i)(-i)^k) \quad (14)$$

as well as $\cos x + \sin x = \sqrt{2} \cos(x - \frac{\pi}{4})$ to compute (11):

$$\begin{aligned}
2a_{\pm} &= \sum_{k=0}^D (1 - \Delta_{\pm}^k) \binom{D}{k} \\
&= \sum_{k=0}^D \left[1 - (\pm)^t (-)^{\frac{1}{2}t(t-1)} (-)^{k(t+1)} (\pm)^k (-)^{\frac{1}{2}k(k-1)} \epsilon_{\pm} \right] \binom{D}{k} \\
&= \sum_{k=0}^D \left[1 - \frac{\rho_{\pm}}{2} \left((1-i) (\pm (-)^{t+1} i)^k + (1+i) (- (\pm) (-)^{t+1} i)^k \right) \right] \binom{D}{k} \\
&= \sum_{k=0}^D \binom{D}{k} - \frac{\rho_{\pm}}{2} (1-i) \sum_{k=0}^D \binom{D}{k} (\pm (-)^{t+1} i)^k \\
&\quad - \frac{\rho_{\pm}}{2} (1+i) \sum_{k=0}^D \binom{D}{k} (- (\pm) (-)^{t+1} i)^k \\
&= 2^D - \frac{\rho_{\pm}}{2} (1-i) (1 + (\pm) (-)^{t+1} i)^D - \frac{\rho_{\pm}}{2} (1+i) (1 - (\pm) (-)^{t+1} i)^D \\
&= 2^D - \frac{\rho_{\pm}}{2} (1-i) (\sqrt{2})^D \exp\left(i \frac{\pm (-)^{t+1} D\pi}{4}\right) \\
&\quad - \frac{\rho_{\pm}}{2} (1+i) (\sqrt{2})^D \exp\left(-i \frac{(\pm) (-)^{t+1} D\pi}{4}\right) \\
&= 2^D - \frac{\rho_{\pm}}{2} 2^{\frac{D}{2}} \left((1-i) \exp\left(\pm i \frac{(-)^{t+1} D\pi}{4}\right) + (1+i) \exp\left(\mp i \frac{(-)^{t+1} D\pi}{4}\right) \right) \\
&= 2^D - \rho_{\pm} 2^{\frac{D}{2}} \left(\cos\left(\frac{\pm (-)^{t+1} D\pi}{4}\right) + \sin\left(\frac{\pm (-)^{t+1} D\pi}{4}\right) \right) \\
&= 2^D - \rho_{\pm} 2^{\frac{D}{2}} \sqrt{2} \cos\left(\frac{(\pm (-)^{t+1} D - 1)\pi}{4}\right)
\end{aligned}$$

We compare this with (13) and get

$$\sqrt{2} \rho_{\pm} \cos\left(\frac{(\pm (-)^{t+1} D - 1)\pi}{4}\right) = 1 \tag{15}$$

which is true for $\rho_{\pm} = -\sqrt{2} \cos\left(\frac{(3 \pm (-)^{t+1} D)\pi}{4}\right)$, because

$$\begin{aligned}
&-2 \cos\left(\frac{(3 \pm (-)^{t+1} D)\pi}{4}\right) \cos\left(\frac{(\pm (-)^{t+1} D - 1)\pi}{4}\right) \\
&= -\cos\left(\frac{(3 \pm (-)^{t+1} D)\pi}{4} - \frac{(\pm (-)^{t+1} D - 1)\pi}{4}\right) \\
&\quad - \cos\left(\frac{(3 \pm (-)^{t+1} D)\pi}{4} + \frac{(\pm (-)^{t+1} D - 1)\pi}{4}\right) \\
&= -\cos \pi - \cos\left(\frac{\pm (-)^{t+1} D\pi}{2} + \frac{\pi}{2}\right) \\
&= 1 + \sin\left(\frac{\pm (-)^{t+1} D\pi}{2}\right) \\
&= 1
\end{aligned}$$

In the first step of the last calculations we used the identity $\cos x \cos y = \cos(x - y) + \cos(x + y)$. In the very last step we plugged in our assumption $D = 2n$ even so that the sine vanishes.

For even dimensions D this is exactly the statement of the proposition:

$$\epsilon_{\pm} = -\sqrt{2}(\pm)^t (-)^{\frac{1}{2}t(t-1)} \cos \frac{(3 \pm (-)^{t+1} D)\pi}{4} \quad (16)$$

To make the construction complete, we turn to the case of odd dimensions. Therefore, we recall that we can construct the gamma-matrices in odd dimension D from the ones in even dimension $d = D - 1$ by adding to the set $\{\gamma_1, \dots, \gamma_d\}$ the matrix $\gamma_D = z\hat{\gamma}$ with $\hat{\gamma} = \gamma_1 \cdots \gamma_d$. The factor $z \in \{1, i\}$ is chosen in such a way that $\gamma_D^2 = \pm \mathbf{1}$ reflecting whether we add a time like or a space like dimension. Without a serious restriction we assume that we add a time like direction, i.e. the amount of time like directions in the d dimensional spacetime is $t - 1$ and the signatures are related via¹ $\sigma = 2t - D = [2(t - 1) - d] + 1 = \sigma_d + 1$. For the square of $\hat{\gamma}$ we have

$$\hat{\gamma}^2 = (-)^{\frac{1}{2}d(d-1)} (-)^{d-(t-1)} \mathbf{1} = (-)^{\frac{1}{2}\sigma_d(\sigma_d-1)} \mathbf{1} = (-)^{\frac{1}{2}\sigma_d} \mathbf{1}. \quad (17)$$

We added a time like direction so that the square of γ_D has to be the identity. This yields that we have to choose $z = (-)^{\frac{1}{4}\sigma_d}$ and take

$$\gamma_D = (-)^{\frac{1}{4}(\sigma-1)} \hat{\gamma}. \quad (18)$$

For this matrix we have by using $1 = zz^*$ and (1)

$$A_{\pm}^{\dagger} \gamma_D A_{\pm} = z A_{\pm}^{\dagger} \gamma_1 \cdots \gamma_d A_{\pm} = z^2 z^* (\pm)^d \gamma_1^* \cdots \gamma_d^* = (-)^{\frac{1}{2}(\sigma-1)} \gamma_D^*. \quad (19)$$

To be consistent with the property $A_{\pm} \gamma_{\mu} A_{\pm} = \pm \gamma_{\mu}^*$ for $1 \leq \mu \leq d$ we have to demand

$$(-)^{\frac{1}{2}(\sigma-1)} = \pm 1. \quad (20)$$

This yields the following choice for the transformation A_{\pm} in the odd dimensional case, in concordance with [1, Table 3]:

$$\begin{aligned} A_+ & \text{ with } \epsilon_+ = + (-) \text{ for } \sigma = 1 (5) \text{ mod } 8 \\ A_- & \text{ with } \epsilon_- = - (+) \text{ for } \sigma = 3 (7) \text{ mod } 8 \end{aligned}$$

REFERENCES

- [1] KLINKER, Frank: *Supersymmetric Killing Structures*, University Leipzig, Germany, PhD., 2003
- [2] SCHERK, J.: Extended supersymmetry and extended supergravity theories. In: *Recent Developments in Gravitation, (Cargèse, 1978), NATO Adv. Study Institutes Series, Ser. B, Vol. 44; Eds. M. Lévy, S. Deser*. New York : Plenum Press, 1979, pp. 479–517

¹This is obviously not possible for Euclidian space time. In that case the construction needs an extra i in front of the matrix γ_D and we have to change the relation between the signatures. This does not have an effect on the result but only on the calculations.