THE SIGNS

SUPPLEMENT TO THE AUTHOR'S DISSERTATION SUPERSYMMETRIC KILLING STRUCTURES

FRANK KLINKER

ABSTRACT. We use the notations introduced in [1, Sect. 1.2] to explain how we derive the signs cf. table 3 therein. The calculations given below are an extended version of the calculations for the Lorentzian case which have their origin in [2].

We denote by t the number of time like directions in the pseudo Riemannian metric of $\mathbb{R}^D$ and by $\sigma = 2t - D$ its signature.

Consider the gamma–matrices γ_μ with $\gamma_\mu^2 = 1$ for $1 \leq \mu \leq t$ and $\gamma_\mu^2 = -1$ for $t + 1 \leq \mu \leq D$.

Furthermore we assume D to be even.

The different transformations are denoted by

$$A_\pm^\dagger \gamma_\mu A_\pm = \pm\gamma_\mu^* \tag{1}$$

$$B^\dagger \gamma^\mu B = (-)^{t+1}\gamma_\mu^\dagger \tag{2}$$

For B we have the explicit description

$$B = \gamma^1 \cdots \gamma^t \tag{3}$$

with

$$B^\dagger = (-)^{\frac{1}{2}t(t-1)} B. \tag{4}$$

From (1) we get

$$(A_\pm^* A_\pm)^T \gamma_\mu (A_\pm A_\pm^*) = A_\pm^T A_\pm^\dagger \gamma_\mu A_\pm A_\pm^* = \pm A_\pm^T \gamma_\mu^* A_\pm^* = \pm(A_\pm^\dagger \gamma_\mu A_\pm)^* = \gamma_\mu$$

which yields $A_\pm A_\pm^* = \epsilon_\pm \mathbf{1}$ or equivalently

$$A_\pm^T = \epsilon_\pm A_\pm. \tag{5}$$

We define the charge conjugation by

$$C_\pm = A_\pm B^*. \tag{6}$$

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Email: frank.klinker@math.tu-dortmund.de.

These matrices satisfy

$$C_\pm^\dagger \gamma_\mu C_\pm = B^T A_\pm^\dagger \gamma_\mu A_\pm B^* = \pm B^T \gamma_\mu^* B^* = \pm(-)^{t+1}\gamma_\mu^T \tag{7}$$

with

$$C_\pm^T = B^\dagger A_\pm^T \overset{(5)(4)}{=} (-)^{\frac{1}{2}t(t-1)}\epsilon_\pm B A_\pm \overset{(1)(3)(6)}{=} (\pm)^t(-)^{\frac{1}{2}t(t-1)}\epsilon_\pm C. \tag{8}$$

This yields the following symmetries for the $\mathfrak{spin}$-invariant morphisms

$$(C_\pm^\dagger \gamma_{\mu_1\cdots\mu_k})^T = (\pm)^t(-)^{\frac{1}{2}t(t-1)}(-)^{k(t+1)}(\pm)^k(-)^{\frac{1}{2}k(k-1)}\epsilon_\pm(C_\pm^\dagger\gamma_{\mu_1\cdots\mu_k}) \tag{9}$$

Compare with [1, Section 2.A, Lemma 2.8] where we called the prefactor $\Delta_\pm^k$ which has the periodicities $\Delta_\pm^{k+2} = -\Delta_\pm^k$ and therefore $\Delta_\pm^{k+4} = \Delta_\pm^k$.

Proposition 1. *The signs $\epsilon_\pm$ in (5) are given by* [1, Table 3] *which is*

σ	0	1	2	3	4	5	6	7
ϵ_+	$+$	$+$	$+$	/	$-$	$-$	$-$	/
ϵ_-	$+$	/	$-$	$-$	$-$	/	$+$	$+$

In the case of even dimension (i.e. even signature) we may explicitly write the sign as

$$\epsilon_\pm = -\sqrt{2}(\pm)^t(-)^{\frac{1}{2}t(t-1)}\cos\frac{(3\pm(-)^{t+1}D)\pi}{4} \tag{10}$$

We count the skew symmetric and symmetric matrices among (9) and get the numbers

$$a_\pm = \frac{1}{2}\sum_{k=0}^{D}(1-\Delta_\pm^k)\binom{D}{k} \tag{11}$$

$$s_\pm = \frac{1}{2}\sum_{k=0}^{D}(1+\Delta_\pm^k)\binom{D}{k} \tag{12}$$

Furthermore we know that

$$2a_\pm = 2^{\frac{D}{2}}(2^{\frac{D}{2}}-1) \quad \text{and} \quad 2s_\pm = 2^{\frac{D}{2}}(2^{\frac{D}{2}}+1). \tag{13}$$

We use the abbreviation $\rho_\pm := (\pm)^t(-)^{\frac{1}{2}t(t-1)}\epsilon_\pm$ and the identities

$$(-)^{\frac{1}{2}k(k-1)} = \frac{1}{2}((1-i)i^k + (1+i)(-i)^k) \tag{14}$$

as well as $\cos x + \sin x = \sqrt{2}\cos(x - \frac{\pi}{4})$ to compute (11):

$$\begin{aligned}
2a_\pm =& \sum_{k=0}^{D}(1-\Delta_\pm^k)\tbinom{D}{k} \\
=& \sum_{k=0}^{D}\Big[1-(\pm)^t(-)^{\frac{1}{2}t(t-1)}(-)^{k(t+1)}(\pm)^k(-)^{\frac{1}{2}k(k-1)}\epsilon_\pm\Big]\tbinom{D}{k} \\
=& \sum_{k=0}^{D}\Big[1-\frac{\rho_\pm}{2}\Big((1-i)\big(\pm(-)^{t+1}i\big)^k+(1+i)\big(-(\pm)(-)^{t+1}i\big)^k\Big)\Big]\tbinom{D}{k} \\
=& \sum_{k=0}^{D}\tbinom{D}{k}-\frac{\rho_\pm}{2}(1-i)\sum_{k=0}^{D}\tbinom{D}{k}\big(\pm(-)^{t+1}i\big)^k \\
&-\frac{\rho_\pm}{2}(1+i)\sum_{k=0}^{D}\tbinom{D}{k}\big(-(\pm)(-)^{t+1}i\big)^k \\
=& 2^D-\frac{\rho_\pm}{2}(1-i)\big(1+(\pm)(-)^{t+1}i\big)^D-\frac{\rho_\pm}{2}(1+i)\big(1-(\pm)(-)^{t+1}i\big)^D \\
=& 2^D-\frac{\rho_\pm}{2}(1-i)(\sqrt{2})^D\exp\big(i\frac{\pm(-)^{t+1}D\pi}{4}\big) \\
&-\frac{\rho_\pm}{2}(1+i)(\sqrt{2})^D\exp\big(-i\frac{(\pm)(-)^{t+1}D\pi}{4}\big) \\
=& 2^D-\frac{\rho_\pm}{2}2^{\frac{D}{2}}\Big((1-i)\exp\big(\pm i\frac{(-)^{t+1}D\pi}{4}\big)+(1+i)\exp\big(\mp i\frac{(-)^{t+1}D\pi}{4}\big)\Big) \\
=& 2^D-\rho_\pm 2^{\frac{D}{2}}\Big(\cos\big(\frac{\pm(-)^{t+1}D\pi}{4}\big)+\sin\big(\frac{\pm(-)^{t+1}D\pi}{4}\big)\Big) \\
=& 2^D-\rho_\pm 2^{\frac{D}{2}}\sqrt{2}\cos\big(\frac{(\pm(-)^{t+1}D-1)\pi}{4}\big)
\end{aligned}$$

We compare this with (13) and get

$$\sqrt{2}\rho_\pm\cos\big(\frac{(\pm(-)^{t+1}D-1)\pi}{4}\big)=1 \tag{15}$$

which is true for $\rho_\pm = -\sqrt{2}\cos\big(\frac{(3\pm(-)^{t+1}D)\pi}{4}\big)$, because

$$\begin{aligned}
&-2\cos\big(\frac{(3\pm(-)^{t+1}D)\pi}{4}\big)\cos\big(\frac{(\pm(-)^{t+1}D-1)\pi}{4}\big) \\
=&-\cos\big(\frac{(3\pm(-)^{t+1}D)\pi}{4}-\frac{(\pm(-)^{t+1}D-1)\pi}{4}\big) \\
&-\cos\big(\frac{(3\pm(-)^{t+1}D)\pi}{4}+\frac{(\pm(-)^{t+1}D-1)\pi}{4}\big)
\end{aligned}$$

$$
\begin{aligned}
&= -\cos\pi - \cos\big(\frac{\pm(-)^{t+1}D\pi}{2} + \frac{\pi}{2}\big) \\
&= 1 + \sin\big(\frac{\pm(-)^{t+1}D\pi}{2}\big) \\
&= 1
\end{aligned}
$$

In the first step of the last calculations we used the identity $\cos x \cos y = \cos(x-y) + \cos(x+y)$. In the very last step we plugged in our assumption $D = 2n$ even so that the sine vanishes.

For even dimensions D this is exactly the statement of the proposition:

$$
\epsilon_\pm = -\sqrt{2}(\pm)^t(-)^{\frac{1}{2}t(t-1)} \cos\frac{(3 \pm (-)^{t+1}D)\pi}{4} \tag{16}
$$

To make the construction complete, we turn to the case of odd dimensions. Therefore, we recall that we can construct the gamma–matrices in odd dimension D from the ones in even dimension $d = D - 1$ by adding to the set $\{\gamma_1, \ldots, \gamma_d\}$ the matrix $\gamma_D = z\hat{\gamma}$ with $\hat{\gamma} = \gamma_1 \cdots \gamma_d$. The factor $z \in \{1, i\}$ is chosen in such a way that $\gamma_D^2 = \pm\mathbf{1}$ reflecting whether we add a time like or a space like dimension. Without a serious restriction we assume that we add a time like direction, i.e. the amount of time like directions in the d dimensional spacetime is $t - 1$ and the signatures are related via[1] $\sigma = 2t - D = [2(t-1) - d] + 1 = \sigma_d + 1$. For the square of $\hat{\gamma}$ we have

$$
\hat{\gamma}^2 = (-)^{\frac{1}{2}d(d-1)}(-)^{d-(t-1)}\mathbf{1} = (-)^{\frac{1}{2}\sigma_d(\sigma_d-1)}\mathbf{1} = (-)^{\frac{1}{2}\sigma_d}\mathbf{1}. \tag{17}
$$

We added a time like direction so that the square of γ_D has to be the identity. This yields that we have to choose $z = (-)^{\frac{1}{4}\sigma_d}$ and take

$$
\gamma_D = (-)^{\frac{1}{4}(\sigma-1)}\hat{\gamma}. \tag{18}
$$

For this matrix we have by using $1 = zz^*$ and (1)

$$
A_\pm^\dagger \gamma_D A_\pm = zA_\pm^\dagger \gamma_1 \cdots \gamma_d A_\pm = z^2 z^* (\pm)^d \gamma_1^* \cdots \gamma_d^* = (-)^{\frac{1}{2}(\sigma-1)}\gamma_D^*. \tag{19}
$$

To be consistent with the property $A_\pm \gamma_\mu A_\pm = \pm\gamma_\mu^*$ for $1 \le \mu \le d$ we have to demand

$$
(-)^{\frac{1}{2}(\sigma-1)} = \pm 1. \tag{20}
$$

This yields the following choice for the transformation $A_\pm$ in the odd dimensional case, in concordance with [1, Table 3]:

$$
\begin{array}{llll}
A_+ & \text{with} \quad \epsilon_+ = +\ (-) & \text{for} \quad \sigma = 1\ (5) & \bmod 8 \\
A_- & \text{with} \quad \epsilon_- = -\ (+) & \text{for} \quad \sigma = 3\ (7) & \bmod 8
\end{array}
$$

[1] This is obviously not possible for Euclidian space time. In that case the construction needs an extra i in front of the matrix γ_D and we have to change the relation between the signatures. This does not have an effect on the result but only on the calculations.

References

[1] Klinker, Frank: *Supersymmetric Killing Structures*, University Leipzig, Germany, PhD., 2003

[2] Scherk, J.: Extended supersymmetry and extended supergravity theories. In: *Recent Developements in Gravitation, (Cargèse, 1978), NATO Adv. Study Institutes Series, Ser. B, Vol. 44; Eds. M. Lévy, S. Deser*. New York : Plenum Press, 1979, pp. 479–517